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BRST treatment of zero modes for the worldline formalism in curved space

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Abstract: One-loop quantities in QFT can be computed in an efficient way using the worldline formalism. The latter rests on the ability of calculating 1D path integrals on the circle. In this paper we give a systematic discussion for treating zero modes on the circle of 1D path integrals for both bosonic and supersymmetric nonlinear sigma models, following an approach originally introduced by Friedan. We use BRST techniques and place a special emphasis on the issue of reparametrization invariance. Various examples are extensively analyzed to verify and test the general set-up. In particular, we explicitly check that the chiral anomaly, which can be obtained by the semiclassical approximation of a supersymmetric 1D path integral, does not receive higher order worldline contributions, as implied by supersymmetry.

Keywords: Sigma Models, BRST Quantization, Anomalies in Field and String Theories.

Contents

1. Introduction

The worldline formalism is an efficient and economical way of calculating Feynman diagrams [1]. It describes the propagation of various relativistic particles by one dimensional (1D) path integrals. Recently we have shown how to extend this method to the case of spin 0 and spin 1/2 particles coupled to background gravity [2, 3]. Other applications run from the calculation of the heat kernel to the calculation of chiral and trace anomalies.

In all these cases, the 1D path integrals are calculated on the circle with a finite propagation time, the proper time. In many circumstances this proper time is integrated over, as it represents the only modulus of the circle. The actions in the path integrals are those of one dimensional nonlinear sigma models. They describe the propagation of particles in a curved background, just like 2D nonlinear sigma models describe the propagation of strings.[1] Sigma models are super-renormalizable in 1D, and their UV structure together with the necessary renormalization conditions (which produce explicit counterterms) have been extensively discussed in the literature using various regularization methods [4, 5, 6].

In this paper we plan to address in a more systematic way the infrared issue related to the treatment of zero modes appearing on the circle and the interplay with the reparametrization invariance of nonlinear sigma models.

[1] The requirement of conformal invariance restricts the possible backgrounds on which the string propagates, but no such a requirement is present for the particle case, at least for spin 0 and 1/2.

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Different ways of treating such zero modes have been developed in the literature. For example in [7] an arbitrary background charge function $\rho(\tau)$ was used to interpolate between various boundary conditions, thus lifting the zero modes in different ways. There it was shown that for models in flat space the effective lagrangian calculated with different background charges ρ differed only by total derivatives, thus producing the same effective action. While this is not causing any particular problem in flat space (total derivatives are even beneficial in certain cases, since they allow to cast the effective action in a more compact form), the naive extension of this method to curved space was seen to introduce noncovariant total derivatives [8, 2], raising questions about the correct use of Riemann normal coordinates to simplify calculations.

A general method for dealing with these zero modes has been developed by Friedan in his treatment of 2D bosonic sigma models [9], and recently employed in the 1D case by Kleinert and Chervyakov [10], where the comparison between the so-called SI (string inspired) propagator and the DBC (Dirichlet boundary conditions) propagator was carried out to show how the former could produce a covariant result as the latter. The string inspired propagator [11] is translational invariant on the worldline, while the DBC one is not.

Here we review and analyze the treatment of the zero modes to clarify it further and resolve some remaining puzzles. We use BRST methods to factor out the zero modes. The BRST symmetry is related to the gauge fixing of a shift symmetry. It is the shift symmetry typical of background field methods [12]. However, in the present case the "background field" is integrated over in the path integral (i.e. it is made dynamical), and thus the shift symmetry must be gauge fixed. An immediate result of this procedure of extracting the zero modes is that different gauges are guaranteed to produce the same effective action, implying that effective lagrangians can only differ by total derivatives, even in curved space. However, the total derivatives will in general be noncovariant. On the other hand, suitable covariant gauges will naturally produce effective lagrangians differing from each other by covariant total derivatives. Various examples are extensively analyzed to test these predictions. In particular, we show how the calculation of the trace of the heat kernel and the related Seeley–DeWitt coefficients, which identify the effective action for a scalar particle in the proper time expansion [13], is achieved with different background charges.

We also extend the zero mode treatment to the supersymmetric case. This is relevant for the worldline description of spin 1/2 fermions coupled to gravity [3]. We use it for an explicit check that the chiral anomaly does not receive higher order worldline contributions. In fact, one may recall that the $N = 1$ supersymmetric nonlinear sigma model was used in [14] to compute the chiral anomaly of a spin 1/2 field. The computation was based on the fact that the chiral anomaly could be identified as the Witten index of the corresponding supersymmetric quantum mechanical model [15]. Supersymmetry implies that higher order worldline contributions should not modify the value of the Witten index, and we test this explicitly using 1D path integrals.

The paper is organized as follows. In section 2 we discuss bosonic nonlinear sigma models. As a test we calculate perturbatively the trace of the heat kernel and the related

Seeley–DeWitt coefficients with different background charges. In section 3 we consider supersymmetric nonlinear sigma models and use them to study worldline corrections to the susy quantum mechanical computation of the chiral anomaly. In section 4 we present our conclusions. For completeness and further clarifications we present the simpler case of a bosonic linear sigma model in appendix A.1. Other appendices include conventions and integrals needed for the computations described in the text.

2. Bosonic nonlinear sigma models

Let us consider the partition function of the 1D nonlinear sigma model

$$Z(\beta) = \oint \mathcal{D}x\, e^{-S[x]}\,, \qquad S[x] = \frac{1}{\beta}\int_0^1 d\tau \left[\frac{1}{2} g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu + \beta^2 V(x)\right] \tag{2.1}$$

where $g_{\mu\nu}$ and V are a metric and a scalar potential defined on target space, which we take to be D dimensional. The path integral is computed with periodic boundary conditions (PBC), i.e. on the circle. We use euclidean time, and it is well-known that periodic boundary conditions then yield the statistical partition function. The circle is just the loop made by the particle in target space. It can be parametrized by $t \in [0, \beta]$, with β the total length of the circle. We also use a rescaled proper time $\tau = t/\beta$ and this rescaling explains the factor $\frac{1}{\beta}$ multiplying the action as well as the β^2 factor in front of the scalar potential.

The partition function $Z(\beta)$ is sometimes called the trace of the heat kernel[2] [13]. It is related to the one-loop effective action of a scalar field with kinetic operator $-\Box + 2V + m^2$ ($\Box$ is the covariant scalar laplacian depending on the metric $g_{\mu\nu}$ and m is the mass of the scalar particle) by an integral over the proper time β

$$\Gamma[g, V] = -\frac{1}{2}\int_0^\infty \frac{d\beta}{\beta}\, e^{-\frac{1}{2}m^2\beta}\, Z(\beta)\,. \tag{2.2}$$

In a perturbative computation zero modes appear. In fact for $g_{\mu\nu} = \delta_{\mu\nu}$ and $V = 0$ the action is invariant under the constant translations $\delta x^\mu(\tau) = \epsilon^\mu$. Hence the volume of target space appears as a factor, just like the volume of a gauge group. Background fields generically break this translation invariance. Nevertheless it is both useful and necessary to extract from the path integral the "collective coordinates" or "center of mass" of the loops $x^\mu(\tau)$, which for simplicity we continue to call zero modes. It is useful, since it allows to produce the partition function as an integral of a partition function density. It is necessary, since in perturbative calculations around the free action one needs to invert the free kinetic term to obtain the perturbative propagator. A general method for treating the zero modes for nonlinear sigma models has been developed by Friedan [9], and employed recently in the 1D case by Kleinert and Chervyakov [10]. Useful references are also [16, 17].

Let us rederive these results and extend them by using an arbitrary background charge $\rho(\tau)$. To extract the zero modes one can proceed as follows.[3] The action of the nonlinear

[2] The operator $e^{-\beta H}$ is called the heat kernel, and the partition function is given by $Z(\beta) = \mathrm{Tr}\, e^{-\beta H} = \oint \mathcal{D}x\, e^{-S[x]}$, where $S[x]$ is the action corresponding to the model with quantum hamiltonian H.

[3] In appendix A.1 we describe the simpler case of a linear sigma model using both Faddeev–Popov and BRST methods.

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sigma model $S[x(\tau)]$ depends on the periodic paths $x^\mu(\tau)$ which describe loops with the topology of a circle in target space. One may introduce a redundant variable x_0^μ by setting $x^\mu(\tau) = x_0^\mu + y^\mu(\tau)$ in the action, $S[x(\tau)] = S[x_0 + y(\tau)]$. Of course this automatically introduces the shift symmetry

$$\begin{aligned} \delta x_0^\mu &= \epsilon^\mu \\ \delta y^\mu(\tau) &= -\epsilon^\mu \end{aligned} \tag{2.3}$$

as in the background field method [12]. We only consider constant x_0^μ so that the shift symmetry requires a constant parameter ϵ^μ. However, contrary to the background field method, we now consider both x_0^μ and $y^\mu(\tau)$ as dynamical variables (i.e. to be integrated over in the path-integral). The shift symmetry is thus promoted to a gauge symmetry and must be gauge fixed since each physical configuration has to be counted only once. To gauge fix we use BRST methods and introduce a constant ghost field η^μ together with the following BRST transformation rules

$$\begin{aligned} \delta x_0^\mu &= \eta^\mu \Lambda \\ \delta y^\mu(\tau) &= -\eta^\mu \Lambda \\ \delta \eta^\mu &= 0\,. \end{aligned} \tag{2.4}$$

To fix a gauge one must also introduce constant nonminimal fields $\bar\eta_\mu, \pi_\mu$ with the BRST rules

$$\delta \bar\eta_\mu = i\pi_\mu \Lambda\,, \qquad \delta \pi_\mu = 0\,. \tag{2.5}$$

Choosing the gauge fixing fermion

$$\Psi = \bar\eta_\mu \int_0^1 d\tau\, \rho(\tau) y^\mu(\tau) \tag{2.6}$$

which depends on the arbitrary function $\rho(\tau)$ normalized to $\int_0^1 d\tau\, \rho(\tau) = 1$ produces the following gauge fixed action

$$\begin{aligned} S_{gf}[x_0, y, \eta, \bar\eta, \pi] &= S[x_0 + y] + \frac{\delta}{\delta\Lambda}\Psi \\ &= S[x_0 + y] + i\pi_\mu \int_0^1 d\tau\, \rho(\tau) y^\mu(\tau) - \bar\eta_\mu \eta^\mu \end{aligned} \tag{2.7}$$

where $\delta/\delta\Lambda$ denotes a BRST variation with the anticommuting parameter Λ removed form the left. In this gauge the ghosts can be trivially integrated out, while the integration over the auxiliary variable π_μ produces a delta function which constrains the fields y^μ to satisfy

$$\int_0^1 d\tau\, \rho(\tau) y^\mu(\tau) = 0\,. \tag{2.8}$$

With this constraint the perturbative kinetic term for the periodic fields $y^\mu(\tau)$, proportional to $d^2/d\tau^2$, can be inverted to obtain the propagator. The BRST symmetry implies that the

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partition function is independent of the gauge parameter ρ. The specific case of $\rho(\tau) = \delta(\tau)$ gives the DBC propagator since $y^\mu(0) = y^\mu(1) = 0$. The case $\rho(\tau) = 1$ gives instead the SI propagator since now the center of mass is absent from the fluctuations y^μ, see e.g. the discussion in [7].

Thus the partition function is independent of ρ and can be expressed as an integral over the zero modes

$$Z(\beta) = \int d^D x_0 \, \frac{\sqrt{g(x_0)}}{(2\pi\beta)^{\frac{D}{2}}} \, z^{(\rho)}(x_0, \beta) \tag{2.9}$$

where the factor $\sqrt{g(x_0)}/(2\pi\beta)^{\frac{D}{2}}$ has been extracted for convenience from the definition of $z^{(\rho)}(x_0, \beta)$. Note however that the density $z^{(\rho)}(x_0, \beta)$ may in general depend on ρ. This can only happen through total derivatives which must then integrate to zero.

Although this is correct, at least formally, there are some practical problems. The constraint arising from the gauge fixing does not have simple transformations rules under change of coordinates (coordinate differences like $y^\mu = x^\mu - x_0^\mu$ do not transform as vectors). This causes some technical problems when one wants to check the explicit covariance of the final result. In particular, one explicitly finds in $z^{(\rho)}(x_0, \beta)$ total derivatives which depend on the choice of ρ and are *not covariant* under change of the coordinates x_0^μ [8, 2]. Let us recall that DBC are related to the calculation of the heat kernel even for non coinciding points, and they are known to give a covariant result. The other propagators related to different background charges produce instead noncovariant total derivatives. A calculation at order β using both the DBC and SI propagators was presented in [2] to explicitly identify the noncovariant total derivative term appearing at that order. For general ρ the expression given in [2] generalizes to

$$z^{(\rho)}(x_0, \beta) = 1 - \beta \left(\frac{1}{24} R + V \right) + \frac{\beta}{2\sqrt{g}} \left(C_\rho + \frac{1}{12} \right) \partial_\mu (\sqrt{g} g^{\alpha\beta} \Gamma_{\alpha\beta}{}^\mu) + O(\beta^2) \tag{2.10}$$

where the precise value of C_ρ is defined later in eq. (2.29). The appearance of noncovariant total derivatives may raise doubts about the use of Riemann normal coordinates which are often used to simplify calculations. In fact, in [8] the assumption of a naive use of Riemann normal coordinates was seen to produce the wrong trace anomaly in 2 and 4 dimensions.

Riemann normal coordinates can nevertheless be used, as showed by Friedan in his discussion of nonlinear sigma models [9]. Friedan noticed that the simple linear shift symmetry (2.3) becomes nonlinear when using Riemann normal coordinates ξ^μ centered at x_0^μ. One should use this nonlinear shift symmetry to correctly perform the gauge fixing in Riemann coordinates. Riemann normal coordinates have the property that they are manifestly covariant under reparametrization of the point x_0^μ. Now the action $S[x_0, \xi(\tau)] \equiv S[x_0 + y(x_0, \xi(\tau))]$ is invariant under the nonlinear shift symmetry

$$\begin{aligned} \delta x_0^\mu &= \epsilon^\mu \\ \delta \xi^\mu(\tau) &= -Q^\mu{}_\nu(x_0, \xi(\tau)) \epsilon^\nu \end{aligned} \tag{2.11}$$

which is a reformulation of (2.3) in these new coordinates. Note that since the origin of the Riemann normal coordinates is shifted, the nonlinear transformation is defined in such

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a way that the new fields $\xi^{\mu\prime} = \xi^\mu + \delta\xi^\mu$ are expressed in Riemann normal coordinates defined around the new origin $x_0^{\mu\prime} = x_0^\mu + \delta x_0^\mu$. The expression $Q^\mu{}_\nu(x_0, \xi)$ can be explicitly calculated and can be found in [9]. We report it here up to the order needed in subsequent calculations

$$\begin{aligned} Q^\mu{}_\nu(x_0,\xi) = {} & \delta^\mu_\nu + \frac{1}{3} R^\mu{}_{\alpha\beta\nu}\,\xi^\alpha\xi^\beta + \frac{1}{12}\nabla_\gamma R^\mu{}_{\alpha\beta\nu}\,\xi^\alpha\xi^\beta\xi^\gamma + \\ & + \left(\frac{1}{60}\nabla_\gamma\nabla_\delta R^\mu{}_{\alpha\beta\nu} - \frac{1}{45} R^\mu{}_{\alpha\beta\lambda} R^\lambda{}_{\gamma\delta\nu}\right)\xi^\alpha\xi^\beta\xi^\gamma\xi^\delta + O(\xi^5)\,. \end{aligned} \tag{2.12}$$

Now we can introduce ghosts and auxiliary fields as usual. The BRST symmetry for the nonlinear shift symmetry is

$$\begin{aligned} \delta x_0^\mu &= \eta^\mu \Lambda \\ \delta\xi^\mu(\tau) &= -Q^\mu{}_\nu(x_0,\xi(\tau))\,\eta^\nu\Lambda \\ \delta\eta^\mu &= 0 \\ \delta\bar\eta_\mu &= i\pi_\mu\Lambda \\ \delta\pi_\mu &= 0\,. \end{aligned} \tag{2.13}$$

It is nilpotent since $Q^\mu{}_\nu(x_0, \xi(\tau))$ satisfies certain relations arising from the abelian nature of the shift symmetry. Using the gauge fermion

$$\Psi = \bar\eta_\mu \int_0^1 d\tau\, \rho(\tau)\xi^\mu(\tau) \tag{2.14}$$

produces the gauge fixed action

$$\begin{aligned} S_{gf}[x_0,\xi,\eta,\bar\eta,\pi] &= S[x_0,\xi] + \frac{\delta}{\delta\Lambda}\Psi \\ &= S[x_0,\xi] + i\pi_\mu \int_0^1 d\tau\,\rho(\tau)\xi^\mu(\tau) - \bar\eta_\mu \int_0^1 d\tau\,\rho(\tau) Q^\mu{}_\nu(x_0,\xi(\tau))\,\eta^\nu\,. \end{aligned} \tag{2.15}$$

The integration over the auxiliary variable π_μ gives again a delta function which constrains the fields ξ^μ to satisfy

$$\int_0^1 d\tau\,\rho(\tau)\xi^\mu(\tau) = 0 \tag{2.16}$$

so that their propagator can be obtained. This constraint has a simple tensorial transformation law under the change of coordinates of x_0^μ, in fact these coordinates transforms as vectors under a reparametrization of the origin x_0^μ. The ghosts now give a nontrivial contribution, i.e. a nontrivial Faddeev–Popov determinant.

The above gauge fixed actions can be used in the path integral. Of course, one also needs to use a path integral measure that is both reparametrization and BRST invariant. This is given by

$$\begin{aligned} \mathcal{D}x &= \prod_{0\le\tau<1} \sqrt{g(x(\tau))}dx(\tau) \longrightarrow dx_0\, d\eta\, d\bar\eta\, d\pi \prod_{0\le\tau<1} \sqrt{g(x(\tau))}dx(\tau) \\ &= dx_0\, d\eta\, d\bar\eta\, d\pi \prod_{0\le\tau<1} \sqrt{g(x_0+y(\tau))}dy(\tau)\,, \end{aligned} \tag{2.17}$$

where we have first added to the sigma model measure the measure for the BRST quartet $x_0^\mu, \eta^\mu, \bar\eta_\mu, \pi_\mu$ which is formally identical to unity (two commuting fields give a volume which is compensated by that of the two anticommuting fields), and then performed the change of variables identified by the "background-quantum" split $x^\mu = x_0^\mu + y^\mu$. We have here used the linear splitting, but we could equally well use the nonlinear one written in terms of the Riemann normal coordinates centered at x_0^μ. In fact the measure is reparametrization invariant, i.e. of the same form in any coordinate system

$$\mathcal{D}x = dx_0\, d\eta\, d\bar\eta\, d\pi \prod_{0\le\tau<1} \sqrt{g(x_0,\xi(\tau))} d\xi(\tau)\,. \tag{2.18}$$

Note that by $g(x_0,\xi(\tau))$ we have indicated the determinant of the metric in Riemann normal coordinates centered at x_0. For future reference we list the expansion of the metric in Riemann coordinates centered at x_0 up to the order needed in later calculations

$$\begin{aligned} g_{\mu\nu}(x_0,\xi) &= g_{\mu\nu}(x_0) + \frac{1}{3} R_{\mu\alpha\beta\nu}(x_0)\xi^\alpha\xi^\beta + \frac{1}{6}\nabla_\gamma R_{\mu\alpha\beta\nu}(x_0)\xi^\alpha\xi^\beta\xi^\gamma + \\ &+ \left(\frac{1}{20}\nabla_\delta\nabla_\gamma R_{\mu\alpha\beta\nu}(x_0) + \frac{2}{45} R_{\mu\alpha\beta}{}^{\sigma} R_{\sigma\gamma\delta\nu}(x_0)\right)\xi^\alpha\xi^\beta\xi^\gamma\xi^\delta + O(\xi^5)\,. \end{aligned} \tag{2.19}$$

It is now useful to introduce additional ghosts to exponentiate the nontrivial part of the measure. We use commuting a^μ and anticommuting b^μ, c^μ ghost fields to reproduce the correct measure [18]

$$\prod_{0\le\tau<1} \sqrt{g(x_0,\xi(\tau))} d\xi(\tau) = \prod_{0\le\tau<1} d\xi(\tau) \int \prod_{0\le\tau<1} da(\tau) db(\tau) dc(\tau)\, e^{-S_{msr}} \tag{2.20}$$

$$S_{msr}[\xi,a,b,c] = \frac{1}{\beta}\int_0^1 d\tau \left[\frac{1}{2} g_{\mu\nu}(x_0,\xi)(a^\mu a^\nu + b^\mu c^\nu)\right]. \tag{2.21}$$

The extra vertices arising from the measure will contribute together with similar vertices from the sigma model action to make the final result finite [19].

We are now ready to re-assemble all parts of the path integral with the zero modes factored out by the nonlinear shift symmetry

$$Z(\beta) = \int dx_0\, d\eta\, d\bar\eta\, d\pi \oint D\xi\, Da\, Db\, Dc\, e^{-S_{gf}[x_0,\xi,\eta,\bar\eta,\pi] - S_{msr}[x_0,\xi,a,b,c]}\,. \tag{2.22}$$

The auxiliary field π^μ can be integrated out to obtain

$$Z(\beta) = \int dx_0\, d\eta\, d\bar\eta \oint D\xi\, Da\, Db\, Dc\, \delta\left(\int_0^1 d\tau\, \rho(\tau)\xi^\mu(\tau)\right) e^{-S_q} \tag{2.23}$$

where

$$S_q = S_{gf}[x_0,\xi,\eta,\bar\eta] + S_{msr}[x_0,\xi,a,b,c] \tag{2.24}$$

with the auxiliary field π_μ eliminated from S_{gf}. Finally, when perturbation around the leading terms in (2.12) and (2.19) is appropriate, one immediately obtains the following perturbative expansion

$$Z(\beta) = \int d^D x_0 \frac{\sqrt{g(x_0)}}{(2\pi\beta)^{\frac{D}{2}}} \langle \exp(-S_q^{(\mathrm{int})})\rangle \tag{2.25}$$

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where the expectation value of the interactions are computed with the propagators

$$\begin{aligned}
\langle \xi^\mu(\tau)\xi^\nu(\sigma)\rangle &= -\beta g^{\mu\nu}(x_0)\mathcal{B}_{(\rho)}(\tau,\sigma) \\
\langle a^\mu(\tau)a^\nu(\sigma)\rangle &= \beta g^{\mu\nu}(x_0)\Delta_{gh}(\tau-\sigma) \\
\langle b^\mu(\tau)c^\nu(\sigma)\rangle &= -2\beta g^{\mu\nu}(x_0)\Delta_{gh}(\tau-\sigma) \\
\langle \eta^\mu \bar{\eta}_\nu\rangle &= -\delta^\mu_\nu \,.
\end{aligned} \tag{2.26}$$

The terms in the measure can be traced back as follows: the factor $\sqrt{g(x_0)}$ is due to the a, b, c ghosts which contain the constant "zero modes", the factor $(2\pi\beta)^{-\frac{D}{2}}$ is the usual free particle measure which corresponds to the determinant of $-\frac{1}{2\beta}\frac{d^2}{d\tau^2}$ on the circle with zero modes excluded. The Green functions appearing in the propagators are as follows. $\mathcal{B}_{(\rho)}(\tau,\sigma)$ is the Green function of the operator $\frac{d^2}{d\tau^2}$ acting on fields constrained by the equation $\int_0^1 d\tau\, \rho(\tau)\xi^\mu(\tau) = 0$. It depends on ρ and satisfies

$$\frac{d^2}{d\tau^2}\mathcal{B}_{(\rho)}(\tau,\sigma) = \delta(\tau-\sigma) - \rho(\tau)\,. \tag{2.27}$$

It is explicitly given by [7]

$$\mathcal{B}_{(\rho)}(\tau,\sigma) = \Delta(\tau-\sigma) - F_\rho(\tau) - F_\rho(\sigma) + C_\rho \tag{2.28}$$

where

$$\begin{aligned}
\Delta(\tau-\sigma) &= \frac{1}{2}|\tau-\sigma| - \frac{1}{2}(\tau-\sigma)^2 - \frac{1}{12} \\
F_\rho(\tau) &= \int_0^1 dx\, \Delta(\tau-x)\rho(x)\,, \qquad C_\rho = \int_0^1 d\tau\, F_\rho(\tau)\rho(\tau)\,.
\end{aligned} \tag{2.29}$$

Clearly it satisfies

$$\int_0^1 d\tau\, \rho(\tau)\, \mathcal{B}_{(\rho)}(\tau,\sigma) = 0\,. \tag{2.30}$$

Note that the auxiliary function $\Delta(\tau-\sigma)$ is the unique translational invariant Green function on the circle (the "string inspired" propagator of [11], which corresponds to $\rho(\tau) = 1$). In the following we will simply write $\mathcal{B}_{(\rho)} = \mathcal{B}$ as no confusion can arise. The Green function for the ghosts is given by the delta function

$$\Delta_{gh}(\tau-\sigma) = \delta(\tau-\sigma) \tag{2.31}$$

but we continue to call it Δ_{gh} as in perturbative calculations it appears in a regulated form.

We are now ready to test this set up. To summarize, the expectations are that the partition function $Z(\beta)$ will not depend on ρ, but that the density $z^{(\rho)}(x_0,\beta)$ will in general be ρ-dependent through total derivatives which integrate to zero. Moreover, using Riemann normal coordinates and the associated nonlinear shift invariance to extract the integral over the zero modes x_0^μ, one expects $z^{(\rho)}(x_0,\beta)$ to be covariant under change of coordinates of x_0^μ, so that the specific total derivatives which may eventually appear will also be covariant.

In the next subsections we explicitly verify, using Riemann normal coordinates, that these total derivatives term are non-zero and covariant also in the presence of external potentials like V.

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2.1 Partition function at 3 loops

We present here the explicit perturbative calculation of the partition function density to order β^2 using Riemann normal coordinates (RNC) and dimensional regularization on the worldline. The nonlinear sigma model in one dimension is super-renormalizable and one needs to choose a specific regularization scheme to compute unambiguously the perturbative expansion. We use dimensional regularization which requires an explicit counterterm $V_{DR} = -\frac{1}{8}R$ to guarantee that the sigma model in (2.1) will have $H = -\frac{1}{2}\Box + V$ as quantum hamiltonian [6]. In the following we will use the rules for dimensional regularization explained in [6]. Dimensional regularization has also been discussed for 1D nonlinear sigma model with infinite proper time in [20].

The partition function density in eq. (2.9) and (2.25) can be expressed in terms of connected worldline graphs as

$$z^{(\rho)}(x_0,\beta) = \langle \exp(-S_q^{(\text{int})})\rangle = \exp(\langle e^{-S_q^{(\text{int})}}\rangle_c - 1) \tag{2.32}$$

where $S_q \equiv S_{gf} + S_{msr}$ is the full quantum action, $\langle\cdots\rangle_c$ denotes connected graphs, and the propagators are given in eq. (2.26).

In order to appreciate the contribution from the FP determinant, we separate the corresponding action

$$S_{FP} = -\bar{\eta}_\mu \int_0^1 d\tau\, \rho(\tau) Q^\mu_\nu(x_0, \xi(\tau))\, \eta^\nu \tag{2.33}$$

from the other contributions, gathered in $\bar{S} \equiv S_q - S_{FP}$. As usual, we organize the interaction terms in such a way that $S_q^{(\text{int})} = S_q - S_{q,2} = S_{q,4} + S_{q,5} + \ldots$, where $S_{q,n} = \bar{S}_n + S_{FP,n}$ gives rise to vevs of order $\beta^{\frac{n}{2}-1}$ (recall that $S_{q,3} = 0$ in RNC, see eqs. (2.12) and (2.19)). Hence

$$\langle e^{-S_q^{(\text{int})}}\rangle_c - 1 = \Big\langle -S_{q,4} - S_{q,6} + \frac{1}{2}S_{q,4}^2\Big\rangle_c + O(\beta^3)\,. \tag{2.34}$$

We denote by V_q the potential which includes the counterterm arising in dimensional regularization

$$V_q = V + V_{DR} = V - \frac{1}{8}R\,. \tag{2.35}$$

We now list the results for the various terms appearing in (2.34) and report in the appendix the expressions and values of the relevant connected one- and two-loop worldline integrals H_i, calculated in dimensional regularization

$$\begin{aligned}
-\langle \bar{S}_4\rangle &= -\frac{\beta}{6}(-H_1 + H_2)R - \beta V_q = \beta\left\{\left(\frac{C_\rho}{3} - \frac{1}{72}\right)R - V_q\right\}\\
-\langle S_{FP,4}\rangle &= \frac{\beta}{3}H_3 R = -\beta\left(\frac{C_\rho}{3} + \frac{1}{36}\right)R\\
-\langle \bar{S}_6\rangle &= \beta^2\left\{\frac{1}{20}(-H_4 + H_5)\,\Box R + \frac{1}{45}(H_4 - H_5)\left[R^2_{\mu\nu} + \frac{3}{2}R^2_{\mu\nu\alpha\beta}\right] + \frac{1}{2}H_1\,\Box V_q\right\}\\
&= \beta^2\Bigg\{\left(2C_\rho^2 - \frac{7}{18}C_\rho - D_\rho - \frac{5}{6}E_\rho + \frac{1}{144}\right)\times
\end{aligned}$$

$$
\begin{aligned}
& \times\left[-\frac{1}{20}\Box R+\frac{1}{45}R^2_{\mu\nu}+\frac{1}{30}R^2_{\mu\nu\alpha\beta}\right]+\frac{1}{2}\left(C_\rho-\frac{1}{12}\right)\Box V_q\Big\} \\
\frac{1}{2}\langle \bar S_4^2\rangle_c &= \beta^2\Big\{\frac{1}{36}(H_6+H_7+2H_8+2H_9-4H_{10}-4H_{11}+2H_{12})R^2_{\mu\nu}+ \\
&\quad +\frac{1}{24}(H_{13}+H_{14}-2H_{15})R^2_{\mu\nu\alpha\beta}\Big\} \\
&= \beta^2\Big\{\left(\frac{1}{18}C_\rho^2+\frac{5}{162}C_\rho+\frac{2}{27}E_\rho-\frac{7}{12960}\right)R^2_{\mu\nu}+ \\
&\quad +\left(-\frac{1}{12}C_\rho^2+\frac{1}{54}C_\rho+\frac{1}{9}E_\rho+\frac{13}{8640}\right)R^2_{\mu\nu\alpha\beta}\Big\} \\
-\langle S_{FP,6}\rangle &= -\beta^2\Big\{\frac{1}{30}H_{16}\,\Box R+\left(\frac{1}{45}H_{16}+\frac{1}{18}H_3^2\right)R^2_{\mu\nu}+\left(\frac{1}{30}H_{16}+\frac{1}{12}H_{17}\right)R^2_{\mu\nu\alpha\beta}\Big\} \\
&= \beta^2\Big\{\left[\frac{1}{10}C_\rho^2-\frac{1}{180}C_\rho-\frac{2}{15}D_\rho-\frac{1}{4320}\right]\Box R+ \\
&\quad +\left[\frac{1}{90}C_\rho^2-\frac{7}{540}C_\rho-\frac{4}{45}D_\rho-\frac{7}{12960}\right]R^2_{\mu\nu}+ \\
&\quad +\left[\frac{1}{60}C_\rho^2-\frac{1}{180}C_\rho+\frac{1}{30}D_\rho-\frac{1}{12}E_\rho-\frac{1}{2880}\right]R^2_{\mu\nu\alpha\beta}\Big\} \\
\frac{1}{2}\langle S^2_{FP,4}\rangle_c &= \frac{\beta^2}{9}H_{17}\,R^2_{\mu\nu}=\frac{\beta^2}{9}\left(C_\rho^2-2D_\rho+E_\rho+\frac{1}{720}\right)R^2_{\mu\nu} \\
\langle S_{FP,4}\,\bar S_4\rangle_c &= \frac{\beta^2}{9}(H_{18}+H_{19}-2H_{20})R^2_{\mu\nu} \\
&= \frac{\beta^2}{9}\left(-2C_\rho^2-\frac{1}{12}C_\rho+3D_\rho-\frac{3}{2}E_\rho-\frac{1}{180}\right)R^2_{\mu\nu}\,. \qquad (2.36)
\end{aligned}
$$

All tensors appearing here are evaluated at x_0.

As a check on these results, note that all contributions from the FP determinant vanish when using DBC. In fact, setting $\rho(\tau)=\delta(\tau)$ one obtains for the ρ dependent coefficients C_ρ, D_ρ, E_ρ the following values $C_{DBC}=-\frac{1}{12}, D_{DBC}=\frac{1}{144}, E_{DBC}=-\frac{1}{180}$. This is expected, since for $\rho(\tau)=\delta(\tau)$ the constraint (2.16) enforces the DBC, i.e. $\xi^\mu(0)=\xi^\mu(1)=0$. This in turn implies that the FP determinant in eq. (2.15) becomes trivial, since $Q^\mu{}_\nu(x_0,0)=\delta^\mu_\nu$, as seen form eq. (2.12). Note also that we have defined the ρ-dependent coefficients to vanish in the SI case, i.e. when $\rho(\tau)=1$.

We can now insert the results (2.36) into (2.34) and (2.32) to obtain the partition function density valid to the order β^2

$$
\begin{aligned}
z^{(\rho)}(x_0,\beta) = \exp\Big\{ & -\beta\left[\frac{1}{24}R+V_q\right]+ \\
& +\beta^2\Big[\left(-\frac{1}{144}C_\rho-\frac{1}{12}D_\rho+\frac{1}{24}E_\rho+\frac{1}{864}\right)\Box R+ \\
& \quad +\left(C_\rho-\frac{1}{12}\right)\left(\frac{1}{48}\Box R+\frac{1}{2}\Box V_q\right)-\frac{1}{720}R^2_{\mu\nu}+\frac{1}{720}R^2_{\mu\nu\alpha\beta}\Big] \\
& +O(\beta^3)\Big\}\,. \qquad (2.37)
\end{aligned}
$$

Expanding the exponent to order β^2, one sees that only covariant total derivatives have ρ-dependent coefficients

$$
\begin{aligned}
z^{(\rho)}(x_0,\beta) &= 1-\beta\left[\frac{1}{24}R+V_q\right]+\\
&\quad+\beta^2\left[\frac{1}{2}\left(\frac{1}{24}R+V_q\right)^2+\frac{1}{720}R^2_{\mu\nu\alpha\beta}-\frac{1}{720}R^2_{\mu\nu}+\right.\\
&\quad\left.+\left(\frac{1}{72}C_\rho-\frac{1}{12}D_\rho+\frac{1}{24}E_\rho-\frac{1}{1728}\right)\Box R+\frac{1}{2}\left(C_\rho-\frac{1}{12}\right)\Box V_q\right]+\\
&\quad+O(\beta^3)\,.
\end{aligned}
\tag{2.38}
$$

Let us discuss some consequences of this formula. We can integrate this density over the zero modes x_0 to obtain the partition function

$$
Z(\beta)=\int d^Dx_0\,\frac{\sqrt{g(x_0)}}{(2\pi\beta)^{\frac{D}{2}}}\left(1+a_1(x_0)\beta+a_2(x_0)\beta^2+\cdots\right)
\tag{2.39}
$$

where $a_n(x_0)$ are the so-called Seeley–DeWitt coefficients. Inserted into (2.2) this gives in turn the effective action of the scalar field with kinetic term $-\Box+2V+m^2$. It is well-known that this proper time expansion of the effective action fails for massless fields. In fact for vanishing mass the damping factor $e^{-\frac{1}{2}m^2\beta}$, which guarantees convergence in eq. (2.2), becomes unity. Nevertheless the Seeley–DeWitt coefficients are still useful in this case as well, as they give the counterterms needed to renormalize the one-loop effective action. Moreover, for conformal fields in D dimensions the coefficient $a_{\frac{D}{2}}$ gives the local trace anomaly [13].

The main points to stress are:

- For the validity of the perturbative calculation of $Z(\beta)$ we have to assume that all external fields describing the interactions should vanish sufficiently fast at infinity. Thus all total derivatives integrate to zero. Therefore gauge independence is verified.

- The covariant local expansions of $z^{(\rho)}(x_0,\beta)$ in Riemann normal coordinates (RNC) are different for different ρ's. The difference is given by total derivatives with coefficients depending on ρ and which are explicitly nonvanishing. From the computation to order β in [10] this could not be evinced. In fact at that order there is no covariant total derivative with the correct dimensions that could possibly contribute. Thus, at order β different ρ's produce the same local expansion. In principle there could have existed hidden relations guaranteeing the same local expansion of $z(x_0,\beta)$ for different ρ's at any order in β. We see explicitly that this is not the case and covariant total derivative may arise.

- One can use these results to compute trace anomalies, which in D dimensions are given by the Seeley–DeWitt coefficient $a_{\frac{D}{2}}$ associated to the corresponding conformal operator. In $D=2$ there cannot be any covariant total derivative contribution to a_1, so the local trace anomaly can be obtained with any ρ. In $D=4$ there is a difference.

However, it only affects the $\Box R$ term, which is a trivial anomaly (it can be canceled by a counterterm). Thus also in this case any ρ is good enough for the computation of the trace anomaly. To be specific, a $D = 4$ conformal scalar needs a potential $V = \frac{\xi}{2} R$, where $\xi = \frac{(D-2)}{4(D-1)} = \frac{1}{6}$. Therefore V_q, which includes the DR counterterm, becomes $V_q = -\frac{1}{24} R$. Inserting V_q into (2.38) produce the following trace anomaly for a conformal scalar field

$$a_2 = \left(-\frac{1}{144}C_\rho - \frac{1}{12}D_\rho + \frac{1}{24}E_\rho + \frac{1}{864}\right)\Box R - \frac{1}{720}R^2_{\mu\nu} + \frac{1}{720}R^2_{\mu\nu\alpha\beta} \tag{2.40}$$

which is the correct value as far as the universal terms are concerned. In fact, only the total derivative term has a ρ–dependent coefficient; the well-known value $\frac{1}{720}$ is recovered with DBC.

- In [10] it was argued that one could reach covariant results for the partition function density also using the "string inspired" propagator in arbitrary coordinates, i.e. through a noncovariant expansion. However the method proposed in section 7 of [10] does not seem to be correct, as far as we understand it. In particular, if one were to include an external potential V in the path integral, there would be no terms in that method that could covariantize it. In any case, these facts may leave some doubts about the correct covariantization obtained using RNC in the presence of external potentials. To make sure that the previously described gauge-fixing in RNC achieves the correct covariantization even with external potentials, in the next section we present a four-loop computation in RNC which tests this issue.

2.2 A look at order β^3

In the last subsection it was pointed out that RNC, with the correct factorization of the zero modes, should produce automatically the correct covariant result for the partition function density $z^{(\rho)}(x_0, \beta)$, also in the presence of terms that are not explicitly constructed from the metric tensor such as an external scalar potential V. In fact, gauge independence of $Z(\beta)$ guarantees that $z^{(\rho)}(x_0, \beta)$ should always contain a universal (ρ-independent) covariant part and a ρ-dependent total derivative, which vanishes upon space-time integration. The ρ-dependent total derivative is in general a noncovariant expression (as seen in eq. (2.10)) if calculated in arbitrary coordinates. It is instead covariant if calculated with RNC.

To make sure that the correct covariantization happens in RNC also in the presence of an external potential V, we compute the terms linear in V that arise at order β^3. They belong to the Seeley–DeWitt coefficient a_3, and are enough to test the correct covariantization of the result. This test is in the same spirit of [8], which employed RNC to a higher perturbative order to discover problems in the computation of the trace anomaly with the SI propagator.

Thus we analyze at order β^3 the partition function density

$$z^{(\rho)}(x_0, \beta) = \left\langle e^{-S_q^{(\mathrm{int})}} \right\rangle \tag{2.41}$$

where, as before, we have included in $S_q^{(\mathrm{int})}$ an arbitrary external scalar potential $V_q = V - \frac{1}{8} R$. It is enough to consider the terms linear in V_q. As a consequence, we only need the perturbative expansion of the ghost action (2.33) to order β^2. We thus only need to calculate the correlators

$$
\begin{aligned}
&\beta \nabla_\alpha V_q \int_0^1 d\tau \int_0^1 d\sigma \Big[\frac{1}{12\beta} \nabla_{\alpha_1} R_{\alpha_2 \mu\nu\alpha_3} \Big\langle \xi^\alpha(\tau)\, \xi^{\alpha_1} \xi^{\alpha_2} \xi^{\alpha_3} (\dot\xi^\mu \dot\xi^\nu + a^\mu a^\nu + b^\mu c^\nu)(\sigma) \Big\rangle + \\
&\qquad\qquad + \frac{1}{12} \nabla_{\alpha_1} R_{\alpha_2\alpha_3} \Big\langle \xi^\alpha(\tau)\, \xi^{\alpha_1} \xi^{\alpha_2} \xi^{\alpha_3}(\sigma) \Big\rangle + \rho(\sigma) \Big] \\
&+ \frac{\beta}{2} \nabla_\alpha \nabla_\beta V_q \int_0^1 d\tau \int_0^1 d\sigma \Big[\frac{1}{6\beta} R_{\alpha_1 \mu\nu\alpha_2} \Big\langle \xi^\alpha \xi^\beta(\tau)\, \xi^{\alpha_1} \xi^{\alpha_2} (\dot\xi^\mu \dot\xi^\nu + a^\mu a^\nu + b^\mu c^\nu)(\sigma) \Big\rangle + \\
&\qquad\qquad + \frac{1}{3} R_{\alpha_1\alpha_2} \Big\langle \xi^\alpha \xi^\beta(\tau)\, \xi^{\alpha_1} \xi^{\alpha_2}(\sigma) \Big\rangle \rho(\sigma) \Big] - \\
&- \frac{\beta}{4!} \nabla_{\alpha_1} \cdots \nabla_{\alpha_4} V_q \int_0^1 d\tau \Big\langle \xi^{\alpha_1} \cdots \xi^{\alpha_4} \Big\rangle - \beta^3 V_q a_2^{(0)}
\end{aligned}
\tag{2.42}
$$

where $a_2^{(0)}$ is the Seeley-DeWitt coefficient computed with vanishing potential V_q. It can be read off directly from (2.38)

$$
a_2^{(0)} = \frac{1}{720}(R^2_{\mu\nu\alpha\beta} - R^2_{\mu\nu}) + \frac{1}{1152} R^2 + \left(\frac{1}{72} C_\rho - \frac{1}{12} D_\rho + \frac{1}{24} E_\rho - \frac{1}{1728} \right) \Box R\,. \tag{2.43}
$$

One thus gets for the relevant terms we wish to consider

$$
\begin{aligned}
&\frac{\beta^3}{6} \nabla_\mu V_q\, \nabla^\mu R (J_1 - J_2 + J_3) + \frac{\beta^3}{6} R_{\mu\nu}\, \nabla^\mu \nabla^\nu V_q (J_4 + J_5 - 2J_6 + 2J_7) + \\
&+ \frac{\beta^3}{12} R \Box V_q\, J_9 (J_{11} - J_{12} + 2J_{10}) - \beta^3 \left[\frac{1}{8} \Box^2 V_q + \frac{1}{12} \nabla_\mu \left(R^{\mu\nu} \nabla_\nu V_q \right) \right] J_8 + \\
&+ \frac{\beta^3}{6} V_q \Box R \left(-\frac{1}{12} C_\rho + \frac{1}{2} D_\rho - \frac{1}{4} E_\rho + \frac{1}{288} \right) + \cdots
\end{aligned}
\tag{2.44}
$$

where J_i are the integrals reported and calculated in appendix C. We finally obtain

$$
z^{(\rho)}(x_0, \beta) = 1 + \cdots + \beta^3 \left(\nabla_\mu Q^\mu - \frac{1}{480} \nabla_\mu V_q \nabla^\mu R + \cdots \right) + \cdots \tag{2.45}
$$

where

$$
\begin{aligned}
Q^\mu \equiv& -\frac{1}{48} \left(C_\rho - \frac{1}{12} \right) R\, \nabla^\mu V_q + \frac{1}{6} \left(-\frac{1}{2} C_\rho^2 + \frac{1}{4} C_\rho + 2D_\rho - \frac{1}{160} \right) R^{\mu\nu}\, \nabla_\nu V_q + \\
&+ \frac{1}{6} \left(-\frac{1}{12} C_\rho + \frac{1}{2} D_\rho - \frac{1}{4} E_\rho + \frac{1}{288} \right) V_q\, \nabla^\mu R - \\
&- \frac{1}{8} \left(C_\rho^2 - \frac{5}{18} C_\rho - \frac{2}{3} E_\rho + \frac{1}{144} \right) \nabla^\mu \Box V_q
\end{aligned}
\tag{2.46}
$$

is the "current" whose divergence gives the total derivative term.

It is then clear that (2.45) yields an unambiguous ρ-independent partition function $Z(\beta)$. For $\rho(\tau) = \delta(\tau)$ one has the DBC propagator which yields directly the correct terms belonging to the local Seeley–DeWitt coefficient a_3 contained in $z^{(\rho)}(x_0, \beta)$ [21]. Others

ρ's must then yield the same local result up to a ρ-dependent covariant total derivative.[4] Thus, we tested successfully the use of RNC to obtain the *correct* covariant results for $Z(\beta)$ also in the presence of external potentials.

3. Supersymmetric nonlinear sigma models

The $N = 1$ supersymmetric nonlinear sigma model is relevant for the worldline description of a spin 1/2 particle coupled to gravity. We have described this worldline approach in [3], where the representation of the one-loop effective action was written in terms of a path integral with periodic boundary conditions (PBC) for the bosonic coordinates x^μ and antiperiodic boundary conditions (ABC) for their supersymmetric partners, the worldline fermions ψ^μ. The worldline fermions play the role of the gamma matrices of the Dirac equation and the ABC are necessary to compute the trace in the spinorial space on which the gamma matrices act. These boundary conditions break worldline supersymmetry. In particular, the antiperiodic fermions ψ^μ do not have any zero mode. The zero modes of the bosons can then be treated as in the previous section.

On the other hand, PBC for both bosons and fermions preserves supersymmetry. The change of boundary conditions from ABC to PBC for the fermions corresponds to an insertion of the chiral matrix γ^5 inside the trace in spinorial space. This gives directly the regulated expression of the Witten index for the nonlinear sigma model [15], which is identified with the chiral U(1) anomaly corresponding to a massless Dirac fermion coupled to gravity [14]. In the present case, the periodic fermions acquire zero modes as well, and it is of interest to consider their factorization. In this section we describe this factorization using directly the action written in components. Superspace methods may also be used, but they do not seem to bring in drastic simplifications.

The $N = 1$ supersymmetric nonlinear sigma model is described by the action

$$S[x,\psi] = \frac{1}{\beta}\int_0^1 d\tau\, \frac{1}{2} g_{\mu\nu}(x)\left(\dot{x}^\mu \dot{x}^\nu + \psi^\mu \frac{D\psi^\nu}{d\tau}\right) \tag{3.1}$$

where $D\psi^\nu/d\tau = \dot{\psi}^\nu + \dot{x}^\lambda \Gamma^\nu_{\lambda\rho}(x)\psi^\rho$ is the worldline time derivative covariantized with the target space connection. We consider PBC for both bosons and fermions, $x^\mu(0) = x^\mu(1)$, $\psi^\mu(0) = \psi^\mu(1)$. The corresponding partition function (Witten index) is then

$$I_W = \oint \mathcal{D}x\, \mathcal{D}\psi\, e^{-S[x,\psi]}\,. \tag{3.2}$$

In the flat limit ($g_{\mu\nu} = \delta_{\mu\nu}$) zero modes appear for both x^μ and ψ^μ, corresponding to the invariances $\delta x^\mu(\tau) = \epsilon^\mu$ and $\delta\psi^\mu(\tau) = \theta^\mu$ for constant ϵ^μ and θ^μ. In curved target space these two invariances are generically broken. Nevertheless we wish to factorize these "would be" zero modes to be able to carry out a perturbative evaluation and at the same time present the partition function as an integral over them. Thus we proceed as in section 2, and introduce extra dynamical gauge variables together with suitable gauge fixing conditions.

[4] The coefficient a_3 is related to the trace anomalies in 6 dimensions [22], which have been calculated also using the quantum mechanical path integral with DBC in [23].

The introduction of extra gauge variables can be obtained by considering the following identities

$$
\begin{aligned}
S[x,\psi] &= S[x_0+y,\psi] = S[x_0+y(x_0,\xi),\psi(\tilde\psi)] \\
&= S[x_0+y(x_0,\xi),\psi(\psi_0+\chi)] \equiv S[x_0,\psi_0,\xi,\chi]
\end{aligned} \tag{3.3}
$$

where we first introduce the new gauge variable x_0^μ, then change coordinates to RNC centered at x_0^μ (this change to new coordinates ξ^μ and $\tilde\psi^\mu$ is specified by the functions $y^\mu(x_0^\nu,\xi^\nu)$ and $\psi^\mu(\tilde\psi^\nu)$), and finally introduce a new gauge variable ψ_0^μ by setting $\tilde\psi^\mu = \psi_0^\mu + \chi^\mu$. We end up with the action $S[x_0,\psi_0,\xi,\chi]$ for the sigma model in RNC

$$
S[x_0,\psi_0,\xi,\chi] = \frac{1}{\beta}\int_0^1 d\tau\, \frac{1}{2} g_{\mu\nu}(x_0,\xi)\left(\dot\xi^\mu\dot\xi^\nu + (\psi_0^\mu+\chi^\mu)\frac{D(\psi_0^\nu+\chi^\nu)}{d\tau}\right) \tag{3.4}
$$

where the metric $g_{\mu\nu}(x_0,\xi)$ in RNC is given in (2.19). This action contains the desired shift gauge symmetries encoded in the following BRST symmetry, suitably extended to two pairs of nonminimal auxiliary fields

$$
\begin{array}{ll}
\delta x_0^\mu = \eta^\mu\Lambda & \delta\xi^\mu(\tau) = -Q^\mu{}_\nu(x_0,\xi(\tau))\,\eta^\nu\Lambda \\
\delta\psi_0^\mu = \gamma^\mu\Lambda & \delta\chi^\mu(\tau) = -\gamma^\nu\Lambda \\
\delta\eta^\mu = 0 & \delta\gamma^\mu = 0 \\
\delta\bar\eta_\mu = i\pi_\mu\Lambda & \delta\pi_\mu = 0 \\
\delta\bar\gamma_\mu = ip_\mu\Lambda & \delta p_\mu = 0\,.
\end{array} \tag{3.5}
$$

The nonlinear bosonic gauge symmetry acting on (x_0^μ,ξ^μ) is just as in the previous section, while the fermionic symmetry acts linearly on the (ψ_0^μ,χ^μ) fields. We gauge fix by choosing the gauge fermion

$$
\Psi = \bar\eta_\mu\int_0^1 d\tau\,\rho(\tau)\xi^\mu(\tau) + \bar\gamma_\mu\int_0^1 d\tau\,\rho(\tau)\chi^\mu(\tau) \tag{3.6}
$$

parametrized by the single function $\rho(\tau)$ normalized to $\int_0^1 d\tau\,\rho(\tau)=1$. This gauge fixing is covariant under reparametrization of x_0^μ, and yields the following gauge fixed action

$$
\begin{aligned}
S_{gf}[x_0,\psi_0,\xi,\chi,\eta,\bar\eta,\gamma,\bar\gamma,\pi,p] \equiv{}& S[x_0,\psi_0,\xi,\chi] + \frac{\delta}{\delta\Lambda}\Psi \\
={}& S[x_0,\psi_0,\xi,\chi] + i\pi_\mu\int_0^1 d\tau\,\rho(\tau)\xi^\mu(\tau) - \\
& -\bar\eta_\mu\int_0^1 d\tau\,\rho(\tau)Q^\mu{}_\nu(x_0,\xi(\tau))\eta^\nu - \\
& -ip_\mu\int_0^1 d\tau\,\rho(\tau)\chi^\mu(\tau) - \bar\gamma_\mu\gamma^\mu\,.
\end{aligned} \tag{3.7}
$$

The ghosts $\gamma^\mu,\bar\gamma_\mu$ can be trivially integrated out, while integration over the lagrange multipliers π_μ and p_μ produces the constraints

$$
\int_0^1 d\tau\,\rho(\tau)\xi^\mu(\tau) = 0\,, \qquad \int_0^1 d\tau\,\rho(\tau)\chi^\mu(\tau) = 0\,. \tag{3.8}
$$

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These constraints permit to invert the kinetic term to find the perturbative propagators for the fields ξ^μ and χ^μ. Thus we end up with the path integral

$$I_W = \int dx_0\, d\psi_0\, d\eta\, d\bar\eta \oint D\xi\, D\chi\, \delta\left(\int \rho\xi^\mu\right) \delta\left(\int \rho\chi^\mu\right) \times \\ \times \exp\left(- S[x_0,\psi_0,\xi,\chi] - S_{FP}[x_0,\xi,\eta,\bar\eta]\right) \tag{3.9}$$

where S_{FP} is the same Faddeev-Popov action of the bosonic model written in (2.33). Note that since we use fermionic fields with curved indices (ψ_0^μ, χ^μ) their nontrivial path integral measure is exactly compensated by the nontrivial measure of their bosonic supersymmetric partners (x_0^μ, ξ^μ). Therefore the complete measure is flat and no extra ghosts are needed in this case, as shown in [3].

The path integral can now be computed. Its value should not depend on β as consequence of supersymmetry [15]. Thus a semiclassical calculation produces already the complete result [14]. However, we have now set up the path integral in such a way that higher loop calculations can be unambiguously performed to test the β independence. We perform this test in the next subsection.

3.1 Order β correction to the chiral anomaly

In this section we consider the perturbative expansion for the $N = 1$ supersymmetric sigma model with periodic boundary conditions on all fields, as described in the previous section. We will explicitly verify the β and ρ independence of the path integral for the chiral anomaly I_W in $D = 2$ and $D = 4$ up to order β (i.e. two loops on the worldline), employing dimensional regularization [3] whenever necessary. Before starting, let us recall that the chiral anomaly I_W can also be written as

$$I_W = \mathrm{Tr}\left((-1)^F\, e^{-\beta Q^2}\right) = \mathrm{Tr}\left(\gamma_5\, e^{\frac{\beta}{2}\nabla\!\!\!\!/\,^2}\right) \tag{3.10}$$

where $Q = \frac{i}{\sqrt{2}}\nabla\!\!\!\!/$ is the supersymmetry charge (the Dirac operator), $H = Q^2 = -\frac{1}{2}\nabla\!\!\!\!/\,^2 = -\frac{1}{2}\Box + \frac{1}{8}R$ the quantum hamiltonian, and $(-1)^F = \gamma_5$ the fermion number operator of the $N = 1$ supersymmetric sigma model. Dimensional regularization preserves supersymmetry without the need of any counterterm [3]: the quantum hamiltonian H must have a potential $\frac{1}{8}R$, obtained from squaring the supercharge Q, and DR produces exactly that potential. Thus the path integral representation of I_W in (3.9) is good as it stands when using dimensional regularization. Other regularization schemes may need counterterms to enforce supersymmetry.

Let us consider the perturbative expansion at fixed x_0 and ψ_0. One obtains the various propagators from the free action

$$S_2 = \frac{1}{2\beta}\, g_{\mu\nu}(x_0) \int_0^1 d\tau\left(\dot\xi^\mu\dot\xi^\nu + \chi^\mu\dot\chi^\nu\right) - \bar\eta_\mu\eta^\mu \tag{3.11}$$

with the fields constrained by (3.8). The propagators for ξ^μ and the ghosts $\eta^\mu, \bar\eta_\mu$ were already described in (2.26), whereas the one for the fermions χ^μ is given by

$$\langle\chi^\mu(\tau)\chi^\nu(\sigma)\rangle = \beta\, g^{\mu\nu}(x_0)\, \mathcal{F}(\tau,\sigma) \tag{3.12}$$

where the Green function

$$\mathcal{F}(\tau,\sigma) = {}^\bullet\Delta(\tau-\sigma) - \int_0^1 d\sigma'\,\rho(\sigma')\,{}^\bullet\Delta(\tau-\sigma') + \int_0^1 d\sigma'\,\rho(\sigma')\,{}^\bullet\Delta(\sigma-\sigma') \tag{3.13}$$

satisfies

$$\begin{aligned} {}^\bullet\mathcal{F}(\tau,\sigma) &= \delta(\tau-\sigma) - \rho(\tau) \\ -\mathcal{F}^\bullet(\tau,\sigma) &= \delta(\tau-\sigma) - \rho(\sigma) \end{aligned} \tag{3.14}$$

(dots on the left/right denote derivatives with respect to the first/second variable) and

$$\int_0^1 d\tau\,\rho(\tau)\,\mathcal{F}(\tau,\sigma) = 0\,. \tag{3.15}$$

With these propagators the chiral anomaly is given by

$$I_W = \int \frac{d^D x_0\, d^D\psi_0}{(2\pi i)^{\frac{D}{2}}}\ \langle e^{-S_q^{(\mathrm{int})}}\rangle \tag{3.16}$$

where $S_q^{(\mathrm{int})} = S_q - S_{q,2}$, $S_q = S + S_{FP}$ is the total action in (3.9), and $\langle 1\rangle = 1$ is the normalization of the (ρ-dependent) path integral. For the sake of compactness, it is useful to rescale $\psi_0 \to \sqrt{\beta}\,\psi_0$, getting $d^D\psi_0 \to \beta^{-D/2}\,d^D\psi_0$ and $\tilde\psi \to \sqrt{\beta}\,\psi_0 + \chi$. Being $\tilde\psi$ of order $\sqrt{\beta}$ we can organize the interaction terms in a systematic way, namely we split $S_q^{(\mathrm{int})} = S_{q,3} + S_{q,4} + \ldots$ with the $S_{q,n} = S_n + S_{FP,n}$ contributions of order $O(\beta^{\frac{n}{2}-1})$. Moreover, in the RNC expansion $S_3 = S_{FP,3} = 0$, as $g_{\mu\nu,\lambda}(x_0) = 0$. Thus, after some simplifying manipulations that exploit index symmetries, the relevant vertices for our purposes read[5]

$$\begin{aligned} S_4 &= \frac{1}{6\beta} R_{\alpha\mu\nu\beta} \int_0^1 d\tau\, \xi^\alpha \xi^\beta \Big(\dot\xi^\mu \dot\xi^\nu + (\sqrt{\beta}\,\psi_0^\mu + \chi^\mu)\dot\chi^\nu \Big) + \\ &\quad + \frac{1}{4\beta} R_{\alpha\mu\nu\lambda} \int_0^1 d\tau\, \xi^\alpha \dot\xi^\mu (\sqrt{\beta}\,\psi_0^\nu + \chi^\nu)(\sqrt{\beta}\,\psi_0^\lambda + \chi^\lambda) \end{aligned} \tag{3.17}$$

$$S_5 = S_{5,1} + \frac{1}{6\beta}\nabla_\beta R_{\alpha\mu\nu\lambda} \int_0^1 d\tau\, \xi^\alpha \xi^\beta \dot\xi^\mu (\sqrt{\beta}\,\psi_0^\nu + \chi^\nu)(\sqrt{\beta}\,\psi_0^\lambda + \chi^\lambda) \tag{3.18}$$

$$\begin{aligned} S_6 &= S_{6,1} + \frac{1}{\beta}\bigg(\frac{5}{72} R_{\lambda\alpha\beta\sigma} R_{\gamma\mu\nu}{}^\sigma + \frac{1}{24} R_{\lambda\alpha\nu\sigma} R_{\beta\mu\gamma}{}^\sigma + \frac{1}{36} R_{\lambda\alpha\beta\sigma} R_{\gamma\nu\mu}{}^\sigma + \frac{1}{16}\ \nabla_\gamma \nabla_\beta R_{\alpha\mu\nu\lambda} \bigg) \times \\ &\quad \times \int_0^1 d\tau\, \xi^\alpha \xi^\beta \xi^\gamma \dot\xi^\mu (\sqrt{\beta}\,\psi_0^\nu + \chi^\nu)(\sqrt{\beta}\,\psi_0^\lambda + \chi^\lambda) \end{aligned} \tag{3.19}$$

where all tensors are evaluated at the point x_0.

[5] $S_{5,1}$ and $S_{6,1}$ are contributions to S_5 and S_6 that come from the cubic and quartic order in the expansion of $g_{\mu\nu}(x)$ in RNC, see (2.19). They are not explicitly reported here since they will not enter in the forthcoming calculations.

It is clear from the Grassmannian nature of the measure that the only terms that can give an *a priori* nonvanishing contribution to (3.16) are those that saturate the measure itself. Hence, in D dimensions only terms with D fermionic zero modes can contribute; in particular

$$\int d^D\psi_0\, \psi_0^{\mu_1}\cdots\psi_0^{\mu_D} = \varepsilon^{\mu_1\cdots\mu_D}\,. \tag{3.20}$$

Before starting our calculation, let us briefly recall how some known results for chiral anomalies in arbitrary dimensions D can be derived from (3.16). First of all, the chiral anomaly is trivially zero when $D = 2k-1$, $k \in \mathbb{N}$, because an odd number of ψ_0 is needed to saturate the measure whereas the fermions always come in pairs. Conversely, if one considers $D = 2k$, it is not difficult to convince oneself [14, 24] that, at the one-loop level, the only nonvanishing contributions come from the k-th power of the vertex $\frac{1}{4}R_{\alpha\mu\nu\lambda}\int_0^1 d\tau\, \xi^\alpha\dot{\xi}^\mu\psi_0^\nu\psi_0^\lambda$; hence, no fermionic propagators are involved. More specifically, the independent terms contributing to the anomalies are of the form

$$\mathrm{Tr}\left[\mathcal{R}^k\right] L_k\,, \qquad \mathcal{R}_{\alpha\beta} \equiv R_{\alpha\beta\mu\nu}\,\psi_0^\mu\psi_0^\nu \tag{3.21}$$

where the integrals L_k are given by

$$L_k = \int_0^1 d\tau_1\cdots\int_0^1 d\tau_k\, {}^\bullet\mathcal{B}(\tau_1,\tau_2){}^\bullet\mathcal{B}(\tau_2,\tau_3)\cdots{}^\bullet\mathcal{B}(\tau_k,\tau_1)\,. \tag{3.22}$$

Integrating by parts and making use of the identity ${}^\bullet\mathcal{B}^\bullet = {}^\bullet\Delta^\bullet$, the latter can be reduced to their string-inspired counterparts

$$\begin{aligned} L_k &= \int_0^1 d\tau_1\cdots\int_0^1 d\tau_k\, {}^\bullet\Delta(\tau_1-\tau_2){}^\bullet\Delta(\tau_2-\tau_3)\cdots{}^\bullet\Delta(\tau_k-\tau_1) \\ &= \begin{cases} \dfrac{2}{(2\pi i)^{2l}}\,\zeta(2l) & k=2l \\ 0 & k=2l+1 \end{cases} \qquad l\in\mathbb{N} \end{aligned} \tag{3.23}$$

where ζ is the Riemann zeta function. Notice that for k odd, both traces and integrals vanish: indeed, gravitational contributions to the chiral U(1) anomaly are present only for spacetime dimensions $D = 4n$, $n \in \mathbb{N}$. The result of this semiclassical approximation is known to be exact [14], and it is independent of the boundary conditions chosen for the quantum fluctuations (see also [24]). Nevertheless, the full path integral (3.9) should be independent of β and ρ. We have explicitly tested this property by doing a two-loop calculation for the simplest cases $D = 2, 4$. For $D = 2$ each worldline graph vanishes, so the proof is quite trivial. Thus, we directly pass to describe the $D = 4$ case.

The two-loop truncation of (3.16) reads

$$I_W = -\int\frac{d^4x_0\, d^4\psi_0}{(2\pi\beta)^2}\exp\left[\left\langle -S_{q,4} - S_{q,6} + \frac{1}{2}(S_{q,4}^2 + S_{q,5}^2) + S_{q,4}S_{q,6} - \frac{1}{3!}S_{q,4}^3\right\rangle_c\right] \tag{3.24}$$

where we have omitted those terms that are explicitly zero; the subscript c refers to connected diagrams. To further simplify this expression, let us make few preliminary considerations. Recalling the splitting $S_q = S + S_{FP}$, we compute

$$\begin{aligned}
\langle S_4\rangle &= \frac{1}{6\beta}R_{\alpha\mu\nu\beta}\int_0^1 d\tau\langle\xi^\alpha\xi^\beta(\dot\xi^\mu\dot\xi^\nu + (\sqrt{\beta}\,\psi_0^\mu + \chi^\mu)\dot\chi^\nu)\rangle + \\
&\quad + \frac{1}{4\beta}R_{\alpha\mu\nu\lambda}\int_0^1 d\tau\langle\xi^\alpha\dot\xi^\mu(\sqrt{\beta}\,\psi_0^\nu + \chi^\nu)(\sqrt{\beta}\,\psi_0^\lambda + \chi^\lambda)\rangle \\
&= \frac{\beta}{6}R_{\alpha\mu\nu\beta}\int_0^1 d\tau\left[g^{\alpha\beta}g^{\mu\nu}\mathcal{B}|_\tau({}^\bullet\mathcal{B}^\bullet + {}^\bullet\mathcal{F})|_\tau + g^{\alpha\nu}g^{\beta\mu}({}^\bullet\mathcal{B}|_\tau)^2\right] \\
&= -\frac{\beta}{2}\,C_\rho\,R\,,
\end{aligned} \tag{3.25}$$

where we have used some of the formulae in appendix B. This term does not contain fermionic zero modes. This is true for all the vertices coming from the ghost action S_{FP}, as well. We also note that $\langle S_6\rangle$ is of order β^2 and contains at most two zero modes. This implies that S_6 can only enter at two-loops through connected terms. Thus, the (one-loop) anomaly is given by

$$\begin{aligned}
I_W^{(0)} &= -\int\frac{d^4x_0\,d^4\psi_0}{(2\pi\beta)^2}\,\frac{1}{2}\,\langle S_4^2\rangle_c = \int\frac{d^4x_0\,d^4\psi_0}{(2\pi)^2}\,\frac{2}{2\cdot 4^2}\,\psi_0^{\alpha_1}\cdots\psi_0^{\alpha_4}\,R_{\alpha_1\alpha_2\mu\nu}\,R_{\alpha_3\alpha_4}{}^{\mu\nu}\,L_2 \\
&= -\frac{1}{768\pi^2}\int d^4x_0\,\varepsilon^{\alpha_1\cdots\alpha_4}R_{\alpha_1\alpha_2\mu\nu}\,R_{\alpha_3\alpha_4}{}^{\mu\nu}\,,
\end{aligned} \tag{3.26}$$

where from eq. (3.23) $L_2 = -\frac{1}{12}$, as expected.

From these preliminaries one recognizes that the order β correction is given by

$$\begin{aligned}
I_W^{(1)} &= \int\frac{d^4x_0\,d^4\psi_0}{(2\pi\beta)^2}\left\{\frac{1}{2}\langle S_4\rangle\langle S_4^2\rangle_c - \frac{1}{2}\langle S_5^2\rangle_c - \langle S_4S_6\rangle_c + \frac{1}{3!}\langle S_4^3\rangle_c + \right. \\
&\qquad\qquad \left. + \frac{1}{2}\langle S_{FP,4}\rangle\langle S_4^2\rangle_c + \frac{1}{2}\langle S_{FP,4}\,S_4^2\rangle_c\right\}.
\end{aligned} \tag{3.27}$$

We first compute the four terms in the first line, leaving the contributions from the FP vertices in the second line for the very end.

The first term can be immediately obtained from (3.25) and (3.26). Using the identities listed in appendix B, it can be written as

$$\frac{1}{2}\int\frac{d^4x_0\,d^4\psi_0}{(2\pi\beta)^2}\,\langle S_4\rangle\langle S_4^2\rangle_c = -\frac{\beta}{384}\int\frac{d^4x_0}{(2\pi)^2}\,C_\rho\,(2K_0 - 8K_2)\,. \tag{3.28}$$

As for the second term, the only way to pick out four zero modes is to square the connection part, the one explicitly reported in (3.18), obtaining

$$\begin{aligned}
\left\langle S_5^2\right\rangle_c^{(4\psi_0)} &= \frac{1}{36}\,\nabla_{\beta_1}R_{\alpha_1\mu_1\lambda_1\lambda_2}\,\nabla_{\beta_2}R_{\alpha_2\mu_2\lambda_3\lambda_4}\,\psi_0^{\lambda_1}\cdots\psi_0^{\lambda_4}\,\times \\
&\quad\times\int_0^1 d\tau_1\int_0^1 d\tau_2\left\langle\xi^{\alpha_1}\xi^{\beta_1}\dot\xi^{\mu_1}\;\xi^{\alpha_2}\xi^{\beta_2}\dot\xi^{\mu_2}\right\rangle \\
&= \frac{\beta^3}{36}\psi_0^{\lambda_1}\cdots\psi_0^{\lambda_4}\Big[\nabla_\alpha R^\alpha{}_{\sigma\lambda_1\lambda_2}\nabla_\beta R^{\beta\sigma}{}_{\lambda_3\lambda_4}(2I_2 - I_3 - I_4) \\
&\qquad + \nabla_\alpha R_{\beta\sigma\lambda_1\lambda_2}(\nabla^\beta R^{\alpha\sigma}{}_{\lambda_3\lambda_4} + \nabla^\alpha R^{\beta\sigma}{}_{\lambda_3\lambda_4})(I_5 - I_6)\Big]\,.
\end{aligned} \tag{3.29}$$

Using the identities in appendix B it becomes

$$-\frac{1}{2}\int \frac{d^4x_0\, d^4\psi_0}{(2\pi\beta)^2}\left\langle S_5^2\right\rangle_c = -\frac{\beta}{2}\int \frac{d^4x_0}{(2\pi)^2}\Big[-\frac{1}{72}\left(K_0+4K_1\right)\,\left(2I_2-I_3-I_4\right)+ \\ +\frac{1}{18}K_2\,\left(4I_2-2I_3-2I_4+3I_5-3I_6\right)+ \\ +\frac{1}{6}K_3\,\left(I_6-I_5\right)\Big]. \tag{3.30}$$

The third term yields

$$\begin{aligned}\left\langle S_4S_6\right\rangle_c^{(4\psi_0)} &= \frac{1}{4}R_{\beta_1\mu_1\alpha_1\alpha_2}\,R_{\rho_2\sigma_2\gamma_2\xi_2\alpha_3\alpha_4}\,\psi_0^{\alpha_1}\cdots\psi_0^{\alpha_4}\times \\ &\quad\times\int_0^1 d\tau_1\int_0^1 d\tau_2\,\langle\xi^{\beta_1}\dot{\xi}^{\mu_1}\xi^{\rho_2}\xi^{\sigma_2}\xi^{\gamma_2}\dot{\xi}^{\xi_2}\rangle \\ &= -\frac{\beta^3}{2}R_{\beta_1\mu_1\alpha_1\alpha_2}\,R_{\rho_2\sigma_2\gamma_2\xi_2\alpha_3\alpha_4}\,\psi_0^{\alpha_1}\cdots\psi_0^{\alpha_4}\times \\ &\quad\times g^{\mu_1\xi_2}(g^{\rho_2\sigma_2}g^{\beta_1\gamma_2}+g^{\rho_2\gamma_2}g^{\beta_1\sigma_2}+g^{\sigma_2\gamma_2}g^{\beta_1\rho_2})\,I_1\end{aligned} \tag{3.31}$$

where the six-index tensor is the one given in the round parenthesis of equation (3.19). After a bit of tensor algebra one gets

$$-\int \frac{d^4x_0\, d^4\psi_0}{(2\pi\beta)^2}\left\langle S_4S_6\right\rangle_c = \frac{\beta}{2}\int \frac{d^4x_0}{(2\pi)^2}\left(-\frac{1}{24}K_0+\frac{2}{9}K_1-\frac{5}{9}K_2+\frac{1}{3}K_3\right)I_1\,. \tag{3.32}$$

Finally, we are left with the fourth and more involved term

$$\left\langle S_4^3\right\rangle_c^{(4\psi_0)} = \left\langle\left(S_{4,1}+S_{4,2}\right)^3\right\rangle_c^{(4\psi_0)} \tag{3.33}$$

where $S_{4,1}$ and $S_{4,2}$ can be read off from (3.17) as the first and second term, respectively. The former may yield only one zero mode and thus it does not give any contribution to (3.33) by itself. The remaining terms give instead

$$\frac{3}{3!}\int \frac{d^4x_0\, d^4\psi_0}{(2\pi\beta)^2}\,\left\langle(S_{4,1})^2S_{4,2}\right\rangle_c = -\frac{\beta}{144}\int\frac{d^4x_0}{(2\pi)^2}\left(4K_2-K_3\right)I_7 \tag{3.34}$$

$$\frac{3}{3!}\int \frac{d^4x_0\, d^4\psi_0}{(2\pi\beta)^2}\,\left\langle S_{4,1}(S_{4,2})^2\right\rangle_c = -\beta\int\frac{d^4x_0}{(2\pi)^2}\times \\ \times\Big[\frac{1}{48}K_0\left(3I_{14}+I_8\right)+ \\ +\frac{1}{24}K_1\left(-2I_{11}+4I_{13}-2I_{12}-I_{10}-I_9\right)+ \\ +\frac{1}{72}K_2\left(6I_{11}-12I_{13}+6I_{12}-6I_8+3I_{10}+3I_9\right)\Big] \tag{3.35}$$

$$\frac{1}{3!}\int \frac{d^4x_0\, d^4\psi_0}{(2\pi\beta)^2}\,\left\langle(S_{4,2})^3\right\rangle_c = \frac{\beta}{16}\int\frac{d^4x_0}{(2\pi)^2}\,K_3\left(2I_{15}-I_{16}-I_{17}\right). \tag{3.36}$$

We now explicitly use the integrals listed in appendix C, so that the four terms computed so far sum up to

$$
\begin{aligned}
-\beta \int \frac{d^4x_0}{(2\pi)^2} \Bigg[& \frac{1}{288}\left(C_\rho + \frac{1}{12}\right) K_0 + \frac{1}{72}\left(-\frac{7}{36}C_\rho - \frac{5}{3}E_\rho + \frac{1}{720}\right) K_1 + \\
& + \frac{1}{72}\left(-\frac{19}{36}C_\rho + \frac{7}{3}E_\rho - \frac{7}{144}\right) K_2 + \frac{1}{72}\left(\frac{5}{36}C_\rho + \frac{1}{3}E_\rho + \frac{13}{720}\right) K_3 \Bigg] . \quad (3.37)
\end{aligned}
$$

This is non zero unless one adopts DBC Feynman rules, for which $C_{\rm DBC} = -1/12$, $E_{\rm DBC} = 1/180$, so that $I_W^{(1)} \propto \int \sum_{l=1}^{3} K_l = 0$ by Bianchi identities. However, the Faddeev-Popov action gives the additional contributions

$$
\begin{aligned}
&\frac{1}{2}\int \frac{d^4x_0\, d^4\psi_0}{(2\pi\beta)^2} \left[\langle S_{FP,4}\rangle \left\langle (S_{4,2})^2\right\rangle_c + \left\langle S_{FP,4}\,(S_{4,2})^2\right\rangle_c\right] = \\
&\qquad = \beta \int \frac{d^4x_0}{(2\pi)^2}\left[\frac{1}{288}\left(C_\rho + \frac{1}{12}\right)(K_0 - 4K_2) + \frac{1}{6}(K_2 - K_1)\, I_{18}\right] \qquad (3.38)
\end{aligned}
$$

which lead to the complete order-β correction

$$
I_W^{(1)} = -\beta \int \frac{d^4x_0}{(2\pi)^2}\frac{1}{72}\left(\frac{5}{36}C_\rho + \frac{1}{3}E_\rho + \frac{13}{720}\right)\sum_{l=1}^{3} K_l = 0\,. \qquad (3.39)
$$

This result shows that one may safely adopt ρ-dependent Feynman rules also in the case of periodic worldline fermions, provided that one appropriately takes care of the zero modes. As we showed, this can be done through the appropriate introduction of auxiliary gauge variables, together with a suitable gauge fixing of the ensuing shift gauge symmetry, which turns the auxiliary gauge variables into the wanted zero modes. As a consequence of this procedure, one must include the corresponding FP determinant, which vanishes only in the DBC case. Indeed, in [25], where a similar calculation has been carried out in DBC as a test for the time slicing regularization, no extra vertices were needed to obtain the correct result.

4. Conclusions

We have discussed in great details the factorization of perturbative zero modes which appear in the worldline approach to one-loop quantities for spin 0 and spin 1/2 particles. We have focused particularly on the issue of reparametrization invariance of the corresponding nonlinear sigma models. As well-known, the calculation of one-loop quantities in the worldline formalism requires to master the path integral for sigma models on the circle. In perturbative computations around flat target space, perturbative zero modes appear. We have described their factorization using BRST methods and employed an arbitrary function $\rho(\tau)$, the so-called background charge, to parametrize different gauges. The arbitrary function $\rho(\tau)$ allows to test for gauge independence and includes as particular cases the two most used methods: *i)* The method based on Dirichlet boundary conditions (DBC), which is directly related to the calculation of the heat kernel. It produces covariant results in arbitrary coordinates also before integrating over zero modes, but it has a propagator that is not translational invariant on the worldline. *ii*) The "string inspired" method, where the constant modes in the Fourier expansion of the quantum fields are factorized. It yields a translational invariant propagator, but when using arbitrary coordinates produces noncovariant total derivatives in the partition function density. The string inspired propagator is simpler and faster for computations on the circle, and the technical problem of the noncovariant total derivatives can be overcome by using Riemann normal coordinates.

The BRST symmetry guarantees the gauge independence of the partition function $Z(\beta)$. From this one deduces that the partition function density $z^{(\rho)}(x_0, \beta)$ should contain a universal ρ-independent covariant part and at most a ρ-dependent total derivative, which vanishes upon space-time integration. The ρ-dependent total derivative is in general a noncovariant expression when calculated in arbitrary coordinates, as exemplified in eq. (2.10). However, it is covariant if calculated using Riemann normal coordinates. The source of this discrepancy is easy to understand: the gauge-fixing constraint is expressed in terms of coordinate differences $x^\mu - x_0^\mu$ and does not transform covariantly as a vector under reparametrization of the background point x_0^μ, but it does once one chooses Riemann coordinates centered at x_0^μ. The gauge fixing in Riemann coordinates is rather subtle. It has originally been studied by Friedan for applications to bosonic 2D nonlinear sigma models [9], and recently employed by Kleinert and Chervyakov [10] in the 1D case. It makes use of a suitable nonlinear shift symmetry.

In this paper we have first addressed some issues left open in [10], and performed extensive tests to check the correctness of the BRST method for extracting the zero modes. Then we have extended this method to the supersymmetric nonlinear sigma model, which is relevant to the description of spin 1/2 particles coupled to gravity. All of our tests have been successful. We can now draw the following conclusions:

- The perturbative calculation of $Z(\beta)$ is gauge independent if the external fields describing the interactions have derivatives vanishing sufficiently fast at infinity. This is a necessary condition for the validity of the perturbative expansions employed in our tests. If these conditions are met, all total derivatives integrate to zero. When the external fields have derivatives which do not vanish fast enough, one may still proceed for example by trying to resum a class of diagrams and make the perturbative expansion of $Z(\beta)$ well-defined. This works in the harmonic oscillator, which has the potential $V(x) = \frac{1}{2}\omega^2 x^2$. In this case we have explicitly checked gauge independence of the partition function up to order $(\beta\omega)^2$.

- The covariant local expansions of $z^{(\rho)}(x_0, \beta)$ in Riemann normal coordinates (RNC) are different for different ρ's. The difference is given by covariant total derivatives with coefficients depending on ρ.

- One can use RNC with any ρ to compute local trace anomalies in $D = 4$. In particular, one can use the string inspired propagator. The total derivatives only affect the $\Box R$ term, which anyway is a trivial anomaly (it can be canceled by a counterterm). The reason why the "string inspired" computation tested in [8] failed is because the

Faddeev-Popov contribution arising form the nonlinear shift symmetry needed for the implementation of RNC was missing.

- The use of RNC yields local covariant results also in the presence of external fields like scalar and vector potentials. We have explicitly verified this in the presence of a scalar potential V. We have performed a higher loop calculation to make sure that the expected results are indeed obtained. It is not obvious how to use arbitrary coordinates to check this property. In particular, the method proposed in section 7 of [10] to achieve a local covariant results in arbitrary coordinates does not seem to be correct, as for example it does not yield the correct covariantization of the higher derivatives of the scalar potential.

- We have used the supersymmetric model with the correct factorization of the zero modes to test successfully for any ρ the β independence of the Witten index, i.e. of the chiral anomaly for a spin 1/2 particle. We have computed the order β corrections the chiral anomaly in $D = 2$ and $D = 4$ and found that they vanish.

The results described here make sure that path integrals for nonlinear sigma model on the circle, with and without supersymmetry, are in good shape both in their UV and IR structure. They can be employed to produce unambiguous results in the worldline formalism with background gravity. The method discussed here could also turn out to be relevant for the extension to the noncommutative case, recently studied in [26] for describing the coupling of $D0$ branes to gravity, and to the free field analysis of the AdS/CFT correspondence [27].

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A. Linear sigma model

Let us consider the linear sigma model

$$S = \frac{1}{\beta}\int_0^1 d\tau \left[\frac{1}{2}\delta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu + \beta^2 V(x)\right] \tag{A.1}$$

and compute perturbatively in β the partition function

$$Z = \mathrm{Tr}\, e^{-\beta H} = \oint Dx\, e^{-S[x]}\,. \tag{A.2}$$

We first try to understand the factorization of the zero mode with some generality. To extract the zero modes one can first introduce a shift symmetry, and then gauge fix it in a suitable way. Using the Faddeev–Popov method one obtains the following chain of identities

$$\begin{aligned} Z &= \oint Dx\, e^{-S[x]} \\ &= \oint Dy\, e^{-S[x_0+y]} \\ &= \oint Dy \int d^D\epsilon\, \delta\Big(\int d\tau\, \rho(\tau) y^\mu_\epsilon(\tau)\Big)\, e^{-S[x_0+y]} \\ &= \int d^D\epsilon \oint Dy_\epsilon\, \delta\Big(\int d\tau\, \rho(\tau) y^\mu_\epsilon(\tau)\Big)\, e^{-S[x_{0\epsilon}+y_\epsilon]} \\ &= \int d^D x_{0\epsilon} \oint Dy\, \delta\Big(\int d\tau\, \rho(\tau) y^\mu(\tau)\Big)\, e^{-S[x_{0\epsilon}+y]} \\ &= \int d^D x_0 \oint Dy\, \delta\Big(\int d\tau\, \rho(\tau) y^\mu(\tau)\Big)\, e^{-S[x_0+y]} \equiv \int \frac{d^D x_0}{(2\pi\beta)^{\frac{D}{2}}}\, z^{(\rho)}(x_0)\,. \end{aligned} \tag{A.3}$$

Here we have first used the linear split $x^\mu(\tau) = x^\mu_0 + y^\mu(\tau)$ for an arbitrary constant x^μ_0, and then the translation invariance of the path integral measure $Dx = Dy$. Then we made use of the shift invariance (typical of the background field method)

$$\begin{aligned} x^\mu_0 &\to x^\mu_{0\epsilon} = x^\mu_0 + \epsilon^\mu \\ y^\mu &\to y^\mu_\epsilon = y^\mu - \epsilon^\mu \end{aligned} \tag{A.4}$$

which leaves invariant the field $x^\mu(\tau)$ since $x^\mu(\tau) = x^\mu_0 + y^\mu(\tau) = x^\mu_{0\epsilon} + y^\mu_\epsilon(\tau)$. Therefore the action $S[x]$ itself remains invariant. One may then use the Faddeev–Popov trick of inserting unity represented as follows

$$\begin{aligned} 1 &= \int d^D\epsilon\, \delta\Big(\int d\tau\, \rho(\tau) y^\mu_\epsilon(\tau)\Big) \\ &= \int d^D\epsilon\, \delta\Big(\int d\tau\, \rho(\tau)(y^\mu(\tau) - \epsilon^\mu)\Big) \\ &= \int d^D\epsilon\, \delta\Big(\int d\tau\, \rho(\tau) y^\mu(\tau) - \epsilon^\mu\Big) \end{aligned} \tag{A.5}$$

where the background charge is normalized to unity, $\int_0^1 d\tau\, \rho(\tau) = 1$. Finally we used that $d^D\epsilon = d^D x_{0\epsilon}$. This formally proves that the final result for the partition function $Z = \int d^D x_0\, (2\pi\beta)^{-\frac{D}{2}} z^{(\rho)}(x_0)$ is gauge invariant, i.e. independent of the choice of the function ρ. However, the density $z^{(\rho)}(x_0)$ can in general depend on ρ through total derivative terms which should vanish upon integration over the zero modes x_0.

One can derive the same result using BRST methods. The action $S[x_0 + y]$ can be considered as a functional of both x^μ_0 and $y^\mu(\tau)$ and is invariant under the shift (A.4). To fix this gauge invariance one can introduce a constant ghost field η^μ for this abelian shift symmetry. The corresponding BRST symmetry transformation rules are

$$\begin{aligned} \delta x^\mu_0 &= \eta^\mu \Lambda \\ \delta y^\mu &= -\eta^\mu \Lambda \\ \delta \eta^\mu &= 0\,. \end{aligned} \tag{A.6}$$

To gauge fix one must also introduce constant nonminimal fields $\bar{\eta}, \pi$ with the BRST rules

$$\begin{aligned} \delta\bar{\eta}_\mu &= i\pi_\mu \Lambda \\ \delta\pi_\mu &= 0\,. \end{aligned} \tag{A.7}$$

Using the gauge fixing fermion

$$\Psi = \bar{\eta}_\mu \int_0^1 d\tau\, \rho(\tau) y^\mu(\tau) \tag{A.8}$$

we obtain the gauge fixed action

$$\begin{aligned} S_{gf} &= S[x_0 + y] + \frac{\delta}{\delta\Lambda}\Psi \\ &= S[x_0 + y] + i\pi_\mu \int_0^1 d\tau\, \rho(\tau) y^\mu(\tau) - \bar{\eta}_\mu \eta^\mu \end{aligned} \tag{A.9}$$

where $\delta/\delta\Lambda$ denotes a BRST variation with the parameter Λ removed form the left. The ghosts can be trivially integrated out while the integration over the auxiliary variable π_μ produces a delta function. Thus, the BRST method reproduces the last line of (A.3), however it make it easier to implement more general gauges (for example to get massive propagators), though they will not be considered here.

To test the previous set-up, let us compute perturbatively in the proper time β the partition function in eq. (A.2). From (A.3) one can write it as

$$\begin{aligned} Z &= \oint Dx\, e^{-S[x]} = \int \frac{d^D x_0}{(2\pi\beta)^{\frac{D}{2}}}\, z^{(\rho)}(x_0) \\ z^{(\rho)}(x_0) &= \langle e^{-S_{\rm int}} \rangle = \exp[\langle e^{-S_{\rm int}} \rangle_c - 1] \end{aligned} \tag{A.10}$$

where $(2\pi\beta)^{-\frac{D}{2}}$ is the well-known normalization of the path integral measure, and the quantum average $\langle e^{-S_{\rm int}} \rangle$ is obtained using the free propagators (i.e. from the action with $V = 0$) of the fields constrained by the gauge fixing. These quantum fields are given by $y^\mu(\tau) = x^\mu(\tau) - x_0^\mu$ with $x_0^\mu = \int_0^1 d\tau \rho(\tau) x^\mu(\tau)$ and their propagator reads

$$\langle y^\mu(\tau) y^\nu(\sigma) \rangle = -\beta \delta^{\mu\nu} \mathcal{B}_{(\rho)}(\tau, \sigma) \tag{A.11}$$

where the Green function $\mathcal{B}_{(\rho)}(\tau, \sigma)$ has already been described in eq. (2.28).

We now aim to compute eq. (A.2) to order β^4 by expanding the interacting action around the constant x_0

$$\begin{aligned} S_{\rm int} &= \frac{1}{\beta} \int_0^1 d\tau\, \beta^2 V(x(\tau)) \\ &= \beta \int_0^1 d\tau \left(\underbrace{V}_{S_4} + \underbrace{y^\mu \partial_\mu V}_{S_5} + \underbrace{\frac{1}{2} y^\mu y^\nu \partial_\mu \partial_\nu V + \cdots}_{S_6} + \underbrace{\frac{1}{6!} y^6 \partial^6 V}_{S_{10}} + \cdots \right) \end{aligned} \tag{A.12}$$

where for S_{10} we used an obvious short-hand notation. Here all vertices are evaluated at x_0 (i.e. one sets $x^\mu(\tau) = x_0^\mu + y^\mu(\tau)$ and expand the action around the constant x_0^μ).

The vertices indicated by S_n contribute effectively like y^n. Considering that S_4 gives only disconnected contributions and that only an even number of y give nonvanishing Wick contractions we obtain

$$\begin{aligned} z^{(\rho)}(x_0) &= \langle e^{-S_{\rm int}} \rangle \\ &= \exp\Big[-\langle S_4 + S_6 + S_8 + S_{10} \rangle + \frac{1}{2}\langle S_5^2 + S_6^2 + 2S_5 S_7 \rangle_c + O(\beta^5) \Big] \end{aligned} \tag{A.13}$$

and thus

$$\begin{aligned} Z = &\frac{1}{(2\pi\beta)^{\frac{D}{2}}} \int d^D x_0 \, \exp \times \\ &\times \Big[-\beta V + \frac{\beta^2}{2} M_1 \Box V - \frac{\beta^3}{2}\Big(\frac{1}{4} M_2 \Box^2 V + M_4 (\partial_\mu V)^2\Big) \\ &\quad + \beta^4 \Big(\frac{1}{48} M_3 \Box^3 V + \frac{1}{4} M_5 (\partial_\mu \partial_\nu V)^2 + \frac{1}{2} M_6 (\partial_\mu V)(\partial^\mu \Box V)\Big) + O(\beta^5) \Big] \end{aligned} \tag{A.14}$$

where the Wick contractions obtained using (A.11) produce the following integrals

$$\begin{aligned} M_1 &= \int_0^1 d\tau \, \mathcal{B}(\tau,\tau) = -\frac{1}{12} + C_\rho \\ M_2 &= \int_0^1 d\tau \, \mathcal{B}^2(\tau,\tau) = \frac{1}{144} + C_\rho^2 - \frac{1}{6} C_\rho + 4C'_\rho \\ M_3 &= \int_0^1 d\tau \, \mathcal{B}^3(\tau,\tau) = -\frac{1}{1728} + C_\rho^3 - \frac{1}{4} C_\rho^2 + \frac{1}{48} C_\rho - 2C'_\rho - 8C''_\rho \\ M_4 &= \int_0^1 d\tau \int_0^1 d\sigma \, \mathcal{B}(\tau,\sigma) = C_\rho \\ M_5 &= \int_0^1 d\tau \int_0^1 d\sigma \, \mathcal{B}^2(\tau,\sigma) = \frac{1}{720} + C_\rho^2 + 2C'_\rho \\ M_6 &= \int_0^1 d\tau \int_0^1 d\sigma \, \mathcal{B}(\tau,\sigma)\mathcal{B}(\sigma,\sigma) = C_\rho^2 - \frac{1}{12} C_\rho + 2C'_\rho \end{aligned}$$

where on top of C_ρ given in eq. (2.29) we have defined $C'_\rho = \int_0^1 d\tau \, F_\rho^2(\tau)$ and $C''_\rho = \int_0^1 d\tau \, F_\rho^3(\tau)$. Recall that the DBC propagator is obtained by setting $\rho(\tau) = \delta(\tau)$ and the SI one by $\rho(\tau) = 1$. In the SI case all these C_ρ's vanish, while the particular values of the integrals for the DBC propagators are as follows

$$M_1 = -\frac{1}{6}, \quad M_2 = \frac{1}{30}, \quad M_3 = -\frac{1}{140}, \quad M_4 = -\frac{1}{12}, \quad M_5 = \frac{1}{90}, \quad M_6 = \frac{1}{60}.$$

Let us now consider the partition function at order β^4 (we rename x_0 by x)

$$\begin{aligned} Z = \frac{1}{(2\pi\beta)^{\frac{D}{2}}} \int d^D x \Big[& 1 - \beta V + \frac{\beta^2}{2}(V^2 + M_1 \Box V) + \\ & + \frac{\beta^3}{24}(-4V^3 - 12 M_1 V \Box V - 3M_2 \Box^2 V - 12 M_4 (\partial_\mu V)^2) + \\ & + \beta^4 \Big(\frac{1}{24} V^4 + \frac{1}{8} M_1^2 (\Box V)^2 + \frac{1}{4} M_1 V^2 \Box V + \frac{1}{2} M_4 V (\partial_\mu V)^2 + \end{aligned}$$

$$+\frac{1}{48}M_3\Box^3 V+\frac{1}{4}M_5(\partial_\mu\partial_\nu V)^2+\frac{1}{2}M_6(\partial_\mu V)(\partial^\mu\Box V)\Big)+$$

$$+O(\beta^5)\Big]\,. \tag{A.15}$$

From this we deduce that

$$\begin{aligned}
\Delta Z &\equiv Z(\text{arbitrary }\rho)-Z(DBC)\\
&=\frac{1}{(2\pi\beta)^{\frac{D}{2}}}\int d^Dx\Big\{\frac{\beta^2}{2}\Big(M_1+\frac{1}{6}\Big)\Box V-\\
&\qquad -\frac{\beta^3}{24}\Big[3(M_2-\frac{1}{30})\Box^2 V+12\Big(M_1+\frac{1}{6}\Big)\partial^\mu(V\partial_\mu V)\Big]+\\
&\qquad +\beta^4\Big[\frac{1}{48}\Big(M_3+\frac{1}{140}\Big)\Box^3 V+\frac{1}{8}\Big(M_1^2-\frac{1}{36}\Big)\partial^\mu(\partial_\mu V\Box V)+\\
&\qquad\quad +\frac{1}{4}\Big(M_5-\frac{1}{90}\Big)\partial^\mu(\partial^\nu V\partial_\mu\partial_\nu V)+\\
&\qquad\quad +\frac{1}{8}\Big(M_2-\frac{1}{30}\Big)\partial^\mu(V\partial_\mu\Box V)+\\
&\qquad\quad +\frac{1}{4}\Big(M_1+\frac{1}{6}\Big)\partial^\mu(V^2\partial_\mu V)\Big]+O(\beta^5)\Big\}
\end{aligned} \tag{A.16}$$

is a total derivative. For potentials V with derivatives vanishing sufficiently fast at infinity (a condition which is necessary for the validity of the perturbation expansion in β) the total derivative can be dropped and all ways of factoring out the zero modes are equivalent.

B. Conventions and identities

We use the following conventions for the curvatures

$$\begin{aligned}
[\nabla_\alpha,\nabla_\beta]V^\mu &= R_{\alpha\beta}{}^\mu{}_\nu V^\nu\\
R_{\mu\nu} &= R_{\lambda\mu}{}^\lambda{}_\nu\,.
\end{aligned} \tag{B.1}$$

The change of coordinates to Riemann normal coordinates is given by

$$y^\mu(x_0,\xi)=\xi^\mu-\sum_{n=2}^{\infty}\frac{1}{n!}\nabla_{\alpha_3}\cdots\nabla_{\alpha_n}\Gamma^\mu_{\alpha_1\alpha_2}(x_0)\xi^{\alpha_1}\cdots\xi^{\alpha_n} \tag{B.2}$$

where the covariant derivatives act on lower indices only.

We have found it convenient to express all the terms in the order-β correction to the chiral anomaly (section 3.1) as combinations of the invariants

$$\begin{aligned}
K_0 &= \varepsilon^{\alpha_1\cdots\alpha_4}\,R^{\mu\nu\rho\sigma}\,R_{\rho\sigma\alpha_1\alpha_2}\,R_{\mu\nu\alpha_3\alpha_4}\\
K_1 &= \varepsilon^{\alpha_1\cdots\alpha_4}\,R^\sigma{}_{\mu\alpha_1}{}^\rho\,R_{\sigma\alpha_2\nu\rho}\,R^{\mu\nu}{}_{\alpha_3\alpha_4}\\
K_2 &= \varepsilon^{\alpha_1\cdots\alpha_4}\,R^\sigma{}_{\mu\alpha_1}{}^\rho\,R_{\sigma\rho\alpha_2\nu}\,R^{\mu\nu}{}_{\alpha_3\alpha_4}\\
K_3 &= \varepsilon^{\alpha_1\cdots\alpha_4}\,R^\sigma{}_{\mu\alpha_1}{}^\rho\,R_{\sigma\nu\rho\alpha_2}\,R^{\mu\nu}{}_{\alpha_3\alpha_4}\,,
\end{aligned}$$

where $K_1 + K_2 + K_3 = 0$ because of Bianchi identities. By using Bianchi identities and integration by parts one can easily prove the following identities

$$\begin{aligned}
\int d^4x\, \varepsilon^{\alpha_1\cdots\alpha_4}\, \nabla_\alpha R^\alpha{}_{\sigma\alpha_1\alpha_2}\, \nabla_\beta R^{\beta\sigma}{}_{\alpha_3\alpha_4} &= \int d^4x \left(-\frac{1}{2}K_0 - 2K_1 + 4K_2\right) \\
\int d^4x\, \varepsilon^{\alpha_1\cdots\alpha_4}\, \Box R^{\beta\sigma}{}_{\alpha_1\alpha_2}\, R_{\beta\sigma\alpha_3\alpha_4} &= \int d^4x\, 4\,(K_3 - K_2) \\
\int d^4x\, \varepsilon^{\alpha_1\cdots\alpha_4}\, R_{\rho\sigma}\, R^\rho_{\mu\alpha_1\alpha_2}\, R^{\sigma\mu}{}_{\alpha_3\alpha_4} &= \int d^4x\, 2\,(K_2 - K_1) \\
\int d^4x\, \varepsilon^{\alpha_1\cdots\alpha_4}\, R_{\alpha_1}{}^\lambda\, R_{\mu\nu\lambda\alpha_2}\, R^{\mu\nu}{}_{\alpha_3\alpha_4} &= \int d^4x \left(\frac{1}{2}K_0 - 2K_2\right) \\
\int d^4x\, \varepsilon^{\alpha_1\cdots\alpha_4}\, R\, R_{\mu\nu\alpha_1\alpha_2}\, R^{\mu\nu}{}_{\alpha_3\alpha_4} &= \int d^4x\, \left(2K_0 - 8K_2\right)
\end{aligned}$$

which have been used to cast the final results in a more compact form.

We now list few useful identities involving the propagators. Recalling the definition of $\Delta(\tau-\sigma)$ in (2.29), we start by defining the quantities

$$\begin{aligned}
C_\rho &= \int_0^1 d\tau_1 \int_0^1 d\tau_2\, \rho(\tau_1)\rho(\tau_2)\Delta(\tau_1-\tau_2) && \left[C_{\rm DBC} = -\frac{1}{12}\right] \\
D_\rho &= \int_0^1 d\tau_1 \int_0^1 d\tau_2 \int_0^1 d\tau_3\, \rho(\tau_1)\rho(\tau_2)\rho(\tau_3)\Delta(\tau_1-\tau_2)\Delta(\tau_1-\tau_3) && \left[D_{\rm DBC} = \frac{1}{144}\right] \\
E_\rho &= \int_0^1 d\tau_1 \int_0^1 d\tau_2\, \rho(\tau_1)\rho(\tau_2)\Delta^2(\tau_1-\tau_2) - \frac{1}{720} && \left[E_{\rm DBC} = \frac{1}{180}\right]
\end{aligned}$$

which all vanish in the string inspired approach ($\rho(\tau) = 1$). For the ρ-dependent propagators we need

$$\begin{aligned}
\mathcal{B}(\tau,\sigma) &= \Delta(\tau-\sigma) - \int_0^1 d\sigma' \rho(\sigma')\, \Delta(\tau-\sigma') - \int_0^1 d\sigma' \rho(\sigma')\, \Delta(\sigma-\sigma') + C_\rho \\
\mathcal{B}|_\tau &\equiv \mathcal{B}(\tau,\tau) = C_\rho - \frac{1}{12} - 2\int_0^1 d\sigma' \rho(\sigma')\, \Delta(\tau-\sigma') \\
{}^\bullet\mathcal{B}(\tau,\sigma) &= {}^\bullet\Delta(\tau-\sigma) - \int_0^1 d\sigma' \rho(\sigma')\, {}^\bullet\Delta(\tau-\sigma') \\
{}^\bullet\mathcal{B}|_\tau &\equiv {}^\bullet\mathcal{B}(\tau,\tau) = -\int_0^1 d\sigma' \rho(\sigma')\, {}^\bullet\Delta(\tau-\sigma') \\
\frac{d}{d\tau}\mathcal{B}|_\tau &= 2\,{}^\bullet\mathcal{B}|_\tau \\
{}^\bullet\mathcal{B}^\bullet(\tau,\sigma) &= {}^\bullet\Delta^\bullet(\tau-\sigma) \\
\mathcal{F}(\tau,\sigma) &= {}^\bullet\Delta(\tau-\sigma) - \int_0^1 d\sigma' \rho(\sigma')\, {}^\bullet\Delta(\tau-\sigma') + \int_0^1 d\sigma' \rho(\sigma')\, {}^\bullet\Delta(\sigma-\sigma') \\
\mathcal{F}|_\tau \equiv \mathcal{F}(\tau,\tau) &= 0 \\
{}^\bullet\mathcal{B}^\bullet(\tau,\sigma) + {}^\bullet\mathcal{F}(\tau,\sigma) &= 1 - \rho(\tau)
\end{aligned}$$

and the identities

$$\int_0^1 d\omega\, {}^\bullet\Delta(\tau-\omega)\, {}^\bullet\Delta(\sigma-\omega) = -\Delta(\tau-\sigma)$$

$$\int_0^1 d\omega\, \Delta(\tau-\omega)\, \Delta(\sigma-\omega) \;=\; -\frac{1}{6}\Delta^2(\tau-\sigma) - \frac{1}{36}\Delta(\tau-\sigma) + \frac{1}{4320}\,.$$

These identities have been used to the express all the worldline integrals, listed in the next appendix, in an economical form, namely as combinations of C_ρ, D_ρ, E_ρ and pure numbers.

C. Integrals

We list here the worldline integrals needed in the main text, evaluated with dimensional regularization whenever necessary. We use the notations $\mathcal{B} \equiv \mathcal{B}(\tau_1,\tau_2)$, $\mathcal{B}|_\tau \equiv \mathcal{B}(\tau,\tau)$, and similarly for its derivatives.

In section 2.1 (partition function at 3 loops), we needed the following integrals

$$\begin{aligned}
H_1 &= \int_0^1 d\tau\, \mathcal{B}|_\tau\ ({}^\bullet\mathcal{B}^\bullet + \Delta_{gh})|_\tau = C_\rho - \frac{1}{12}\\
H_2 &= \int_0^1 d\tau\, {}^\bullet\mathcal{B}|_\tau^2 = -C_\rho\\
H_3 &= \int_0^1 d\tau\, \rho(\tau)\ \mathcal{B}|_\tau = -C_\rho - \frac{1}{12}\\
H_4 &= \int_0^1 d\tau\, \mathcal{B}|_\tau^2\ ({}^\bullet\mathcal{B}^\bullet + \Delta_{gh})|_\tau = C_\rho^2 - \frac{5}{18}C_\rho - \frac{2}{3}E_\rho + \frac{1}{144}\\
H_5 &= \int_0^1 d\tau\, \mathcal{B}|_\tau\ {}^\bullet\mathcal{B}|_\tau^2 = -C_\rho^2 + \frac{1}{9}C_\rho + D_\rho + \frac{1}{6}E_\rho\\
H_6 &= \int_0^1 d\tau_1 \int_0^1 d\tau_2\, \mathcal{B}|_1\ \mathcal{B}|_2\ ({}^\bullet\mathcal{B}^{\bullet 2} - \Delta_{gh}^2) = -C_\rho^2 + \frac{7}{18}C_\rho + \frac{4}{3}E_\rho - \frac{1}{144}\\
H_7 &= \int_0^1 d\tau_1 \int_0^1 d\tau_2\, \mathcal{B}^2\ ({}^\bullet\mathcal{B}^\bullet + \Delta_{gh})|_1 = C_\rho^2 - \frac{1}{18}C_\rho - \frac{1}{3}E_\rho + \frac{1}{720}\\
H_8 &= \int_0^1 d\tau_1 \int_0^1 d\tau_2\, {}^\bullet\mathcal{B}|_1\ {}^\bullet\mathcal{B}^\bullet\ \mathcal{B}^\bullet|_2\ \mathcal{B} = C_\rho^2 - \frac{1}{12}C_\rho - D_\rho\\
H_9 &= \int_0^1 d\tau_1 \int_0^1 d\tau_2\, {}^\bullet\mathcal{B}|_1\ {}^\bullet\mathcal{B}\ \mathcal{B}^\bullet|_2\ \mathcal{B}^\bullet = C_\rho^2 + \frac{1}{36}C_\rho - D_\rho + \frac{1}{6}E_\rho\\
H_{10} &= \int_0^1 d\tau_1 \int_0^1 d\tau_2\, {}^\bullet\mathcal{B}|_1\ {}^\bullet\mathcal{B}^\bullet\ \mathcal{B}|_2\ \mathcal{B}^\bullet = C_\rho^2 - \frac{1}{6}C_\rho - D_\rho - \frac{1}{2}E_\rho\\
H_{11} &= \int_0^1 d\tau_1 \int_0^1 d\tau_2\, {}^\bullet\mathcal{B}|_1\ {}^\bullet\mathcal{B}\ \mathcal{B}\ ({}^\bullet\mathcal{B}^\bullet + \Delta_{gh})|_2 = -C_\rho^2 + \frac{1}{24}C_\rho + \frac{1}{2}D_\rho + \frac{1}{4}E_\rho\\
H_{12} &= \int_0^1 d\tau_1 \int_0^1 d\tau_2\, \mathcal{B}|_1\ {}^\bullet\mathcal{B}^2\ ({}^\bullet\mathcal{B}^\bullet + \Delta_{gh})|_2\ \ = -C_\rho^2 + \frac{7}{36}C_\rho + D_\rho + \frac{1}{6}E_\rho - \frac{1}{144}\\
H_{13} &= \int_0^1 d\tau_1 \int_0^1 d\tau_2\, \mathcal{B}^2\ ({}^\bullet\mathcal{B}^{\bullet 2} - \Delta_{gh}^2) = -C_\rho^2 + \frac{1}{4}C_\rho + E_\rho + \frac{1}{120}\\
H_{14} &= \int_0^1 d\tau_1 \int_0^1 d\tau_2\, {}^\bullet\mathcal{B}^2\ \mathcal{B}^{\bullet 2} = C_\rho^2 - 2D_\rho + E_\rho + \frac{1}{80}\\
H_{15} &= \int_0^1 d\tau_1 \int_0^1 d\tau_2\, {}^\bullet\mathcal{B}\ {}^\bullet\mathcal{B}^\bullet\ \mathcal{B}^\bullet\ \mathcal{B} = C_\rho^2 - \frac{7}{72}C_\rho - D_\rho - \frac{1}{3}E_\rho - \frac{11}{1440}
\end{aligned}$$

$$
\begin{aligned}
H_{16} &= \int_0^1 d\tau\, \rho(\tau)\; \mathcal{B}|_\tau^2 = -3C_\rho^2 + \frac{1}{6}C_\rho + 4D_\rho + \frac{1}{144} \\
H_{17} &= \int_0^1 d\tau_1 \int_0^1 d\tau_2\, \rho(\tau_1)\rho(\tau_2)\; \mathcal{B}^2 = C_\rho^2 - 2D_\rho + E_\rho + \frac{1}{720} \\
H_{18} &= \int_0^1 d\tau_1 \int_0^1 d\tau_2\, \rho(\tau_2)\; \mathcal{B}^2\; ({}^\bullet\mathcal{B}^\bullet + \Delta_{gh})|_1 = -C_\rho^2 + \frac{1}{36}C_\rho + D_\rho + \frac{1}{6}E_\rho + \frac{1}{720} \\
H_{19} &= \int_0^1 d\tau_1 \int_0^1 d\tau_2\, \rho(\tau_2)\; {}^\bullet\mathcal{B}^2\; \mathcal{B}|_1 = C_\rho^2 - \frac{5}{36}C_\rho - D_\rho - \frac{5}{6}E_\rho - \frac{1}{144} \\
H_{20} &= \int_0^1 d\tau_1 \int_0^1 d\tau_2\, \rho(\tau_2)\; \mathcal{B}\; {}^\bullet\mathcal{B}^2\; {}^\bullet\mathcal{B}|_1 = C_\rho^2 - \frac{1}{72}C_\rho - \frac{3}{2}D_\rho + \frac{5}{12}E_\rho\,.
\end{aligned}
$$

In the 4 loop calculation of section 2.2 we made use of the following integrals

$$
\begin{aligned}
J_1 &= \int_0^1 d\tau_1 \int_0^1 d\tau_2\, \mathcal{B}\; \mathcal{B}|_2\; ({}^\bullet\mathcal{B}^\bullet + \Delta_{gh})|_2 = C_\rho^2 - \frac{5}{36}C_\rho - \frac{1}{3}E_\rho \\
J_2 &= \int_0^1 d\tau_1 \int_0^1 d\tau_2\, \mathcal{B}\; ({}^\bullet\mathcal{B}^2)|_2 = -C_\rho^2 + \frac{1}{72}C_\rho + \frac{1}{2}D_\rho + \frac{1}{12}E_\rho \\
J_3 &= \int_0^1 d\tau_1 \int_0^1 d\tau_2\, \rho(\tau_2)\; \mathcal{B}\; \mathcal{B}|_2 = -2C_\rho^2 + 2D_\rho \\
J_4 &= \int_0^1 d\tau_1 \int_0^1 d\tau_2\, \mathcal{B}^2\; ({}^\bullet\mathcal{B}^\bullet + \Delta_{gh})|_2 = C_\rho^2 - \frac{1}{18}C_\rho - \frac{1}{3}E_\rho + \frac{1}{720} \\
J_5 &= \int_0^1 d\tau_1 \int_0^1 d\tau_2\, \mathcal{B}^{\bullet 2}\; \mathcal{B}|_2 = -C_\rho^2 + \frac{7}{36}C_\rho + D_\rho + \frac{1}{6}E_\rho - \frac{1}{144} \\
J_6 &= \int_0^1 d\tau_1 \int_0^1 d\tau_2\, \mathcal{B}\; \mathcal{B}^\bullet\; {}^\bullet\mathcal{B}|_2 = -C_\rho^2 + \frac{1}{24}C_\rho + \frac{1}{2}D_\rho + \frac{1}{4}E_\rho \\
J_7 &= \int_0^1 d\tau_1 \int_0^1 d\tau_2\, \rho(\tau_2)\; \mathcal{B}^2 = -C_\rho^2 + \frac{1}{36}C_\rho + D_\rho + \frac{1}{6}E_\rho + \frac{1}{720} \\
J_8 &= \int_0^1 d\tau\, \mathcal{B}^2|_\tau = C_\rho^2 - \frac{5}{18}C_\rho - \frac{2}{3}E_\rho + \frac{1}{144}\;. \\
J_9 &= \int_0^1 d\tau\, \mathcal{B}|_\tau = C_\rho - \frac{1}{12} \\
J_{10} &= \int_0^1 d\tau\, \rho(\tau)\; \mathcal{B}|_\tau = -C_\rho - \frac{1}{12} \\
J_{11} &= \int_0^1 d\tau\, \mathcal{B}|_\tau\; ({}^\bullet\mathcal{B}^\bullet + \Delta_{gh})|_\tau = C_\rho - \frac{1}{12} \\
J_{12} &= \int_0^1 d\tau\, {}^\bullet\mathcal{B}^2|_\tau = -C_\rho\,.
\end{aligned}
$$

Finally, the integrals needed in the calculation of the order-β correction to the chiral anomaly in $D = 4$ (section 3.1) are given by

$$
\begin{aligned}
I_1 &= \int_0^1 d\tau_1 \int_0^1 d\tau_2\, \mathcal{B}{}^\bullet\mathcal{B}^\bullet\mathcal{B}|_2 = \frac{5}{36}C_\rho + \frac{1}{3}E_\rho - \frac{1}{144} \\
I_2 &= \int_0^1 d\tau_1 \int_0^1 d\tau_2\, \mathcal{B}|_1{}^\bullet\mathcal{B}^\bullet\mathcal{B}|_2 = -2I_3
\end{aligned}
$$

$$
\begin{aligned}
I_3 &= \int_0^1 d\tau_1 \int_0^1 d\tau_2\, {}^\bullet\mathcal{B}|_1 \mathcal{B}^\bullet\mathcal{B}|_2 = \frac{1}{36}C_\rho + \frac{1}{6}E_\rho \\
I_4 &= \int_0^1 d\tau_1 \int_0^1 d\tau_2\, \mathcal{B}|_1 {}^\bullet\mathcal{B}^\bullet\mathcal{B}|_2 = 4I_3 \\
I_5 &= \int_0^1 d\tau_1 \int_0^1 d\tau_2\, {}^\bullet\mathcal{B}\mathcal{B}^\bullet\mathcal{B} = -\frac{1}{9}C_\rho - \frac{1}{6}E_\rho + \frac{1}{360} \\
I_6 &= \int_0^1 d\tau_1 \int_0^1 d\tau_2\, \mathcal{B}\mathcal{B}^\bullet\mathcal{B}^\bullet = -2I_5 \\
I_7 &= \int_0^1 d\tau_1 \int_0^1 d\tau_2 \int_0^1 d\tau_3\, (\mathcal{B}^\bullet\mathcal{F}^\bullet)_{(1,2)}\mathcal{B}_{(2,3)}{}^\bullet\mathcal{B}_{(3,1)} = I_1 \\
I_8 &= \int_0^1 d\tau_1 \int_0^1 d\tau_2 \int_0^1 d\tau_3\, \mathcal{B}|_1 {}^\bullet\mathcal{F}_{(1,2)}({}^\bullet\mathcal{B}\mathcal{B}^\bullet)_{(2,3)} = -\frac{2}{9}C_\rho - \frac{1}{3}E_\rho \\
I_9 &= \int_0^1 d\tau_1 \int_0^1 d\tau_2 \int_0^1 d\tau_3\, (\mathcal{B}^\bullet\mathcal{F})_{(1,2)}{}^\bullet\mathcal{B}_{(2,3)}{}^\bullet\mathcal{B}_{(3,1)} = -\frac{1}{6}C_\rho - \frac{1}{2}E_\rho + \frac{1}{180} \\
I_{10} &= \int_0^1 d\tau_1 \int_0^1 d\tau_2 \int_0^1 d\tau_3\, (\mathcal{B}^{\bullet\bullet}\mathcal{F})_{(1,2)}\mathcal{B}_{(2,3)}{}^\bullet\mathcal{B}_{(3,1)} = -\frac{1}{36}C_\rho - \frac{1}{6}E_\rho - \frac{1}{720} \\
I_{11} &= \int_0^1 d\tau_1 \int_0^1 d\tau_2 \int_0^1 d\tau_3\, ({}^\bullet\mathcal{B}^\bullet + {}^\bullet\mathcal{F})|_1 \mathcal{B}^\bullet_{(1,2)}\mathcal{B}_{(2,3)}{}^\bullet\mathcal{B}_{(3,1)} = \frac{1}{18}C_\rho + \frac{1}{3}E_\rho \\
I_{12} &= \int_0^1 d\tau_1 \int_0^1 d\tau_2 \int_0^1 d\tau_3\, \mathcal{B}|_1 {}^\bullet\mathcal{B}_{(1,2)}{}^\bullet\mathcal{B}^\bullet_{(2,3)}\mathcal{B}^\bullet_{(3,1)} = -\frac{1}{12}C_\rho + \frac{1}{144} \\
I_{13} &= \int_0^1 d\tau_1 \int_0^1 d\tau_2 \int_0^1 d\tau_3\, {}^\bullet\mathcal{B}|_1 \mathcal{B}_{(1,2)}{}^\bullet\mathcal{B}^\bullet_{(2,3)}\mathcal{B}^\bullet_{(3,1)} = -\frac{1}{36}C_\rho - \frac{1}{6}E_\rho \\
I_{14} &= \int_0^1 d\tau_1 \int_0^1 d\tau_2 \int_0^1 d\tau_3\, ({}^\bullet\mathcal{B}\mathcal{B}^\bullet)_{(1,2)}({}^\bullet\mathcal{B}\mathcal{B}^\bullet)_{(2,3)} = -\frac{1}{36}C_\rho - \frac{1}{6}E_\rho + \frac{1}{144} \\
I_{15} &= \int_0^1 d\tau_1 \int_0^1 d\tau_2 \int_0^1 d\tau_3\, (\mathcal{F}\mathcal{B}^\bullet)_{(1,2)}\mathcal{B}^\bullet_{(2,3)}\mathcal{B}^\bullet_{(3,1)} = -\frac{1}{36}C_\rho - \frac{1}{6}E_\rho + \frac{1}{360} \\
I_{16} &= \int_0^1 d\tau_1 \int_0^1 d\tau_2 \int_0^1 d\tau_3\, (\mathcal{F}\mathcal{B})_{(1,2)}{}^\bullet\mathcal{B}^\bullet_{(2,3)}\mathcal{B}^\bullet_{(3,1)} = -\frac{5}{36}C_\rho - \frac{1}{3}E_\rho + \frac{1}{360} \\
I_{17} &= \int_0^1 d\tau_1 \int_0^1 d\tau_2 \int_0^1 d\tau_3\, (\mathcal{F}^\bullet\mathcal{B}^\bullet)_{(1,2)}\mathcal{B}^\bullet_{(2,3)}\mathcal{B}_{(3,1)} = \frac{1}{18}C_\rho + \frac{1}{3}E_\rho - \frac{1}{720} \\
I_{18} &= \int_0^1 d\tau_1 \int_0^1 d\tau_2 \int_0^1 d\tau_3\, \rho(\tau_1)\mathcal{B}^\bullet_{(1,2)}\mathcal{B}^\bullet_{(2,3)}\mathcal{B}_{(3,1)} = \frac{1}{36}C_\rho + \frac{1}{6}E_\rho + \frac{1}{720}\,.
\end{aligned}
$$

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